On initial value problem for one-dimension semi linear Eliptic Partial Differential Equation with null condition

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Abstract : In this research paper we work on the initial-boundary value problem on $R^+ \times R^+$ for one-dimension systems of semilinear eliptic partial differential equations with null conditions. It is show that in homogeneous Dirichlet or Neumann boundary values at sufficiently small initial data, classical solutions always globally exist.

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I. INTRODUCTION

The global existence of classical solutions for Cauchy problems of one-dimension semi linear wave equations satisfying null conditions with small initial data is proved by Luli, Yang and Yu [8]and this former result is obtained Nakamura [11]. This result can be viewed[8] as a one dimensional and semi linear analogue of Christo doulou and Klainerman'spioneering works for the global existence of classical solutions forn online ar wave equations with null conditions in three space dimensions[2,5], and of Alinhac'sglobal existence result for the case of two space dimensions [1] (see also[3,12,7] for some thorough studies in the 2-Dcase). The proof sin[2,5,1] are based on the time ofline arwaves, in one space dimension waves do not decay in time.

In this Research paper ,we induce the to the initial-boundary value problem for one-dimension semi linear wave equations with null conditions on Elliptic Partial differential Equation. Specifically we consider the following initial-boundary value problem for one-dimension semi linear wave equations

$$\begin{cases} u_{tt} - u_{xx}, & t > 0, x > 0 \\ (Bu)(t,0) = 0, & t \geq 0 \\ t = 0: u = u_0(x), u_t = u_1(x), x > 0 \end{cases}$$

where the function $u = u(t,x): R^+ \times R^+ \to F: R^n \times R^n \to R^n$ is given smooth and vector valued function , the boundary operator

$$B(u)t(0) = \begin{cases} u(t,0), & Dirichlet\ type \\ u_x(t,0), & Neumann\ type \end{cases}$$

We will always assume that the nonlinearity in the system (1.1) is quadratic and satisfies the global existence of result with null condition [4,6,9,10] that is

$$F^i(u_t, u_x) = C_{ijk} u_\xi^j u_\eta^k \ , \ 1 \leq i \leq n \qquad (1.3)$$

Where the null coordinates

$\xi = \frac{t+x}{2}, \quad \eta = \frac{t-x}{2}$ (1.4)

The null vector fields

$\partial_\xi = \partial_t + \partial_x \ , \partial_\eta = \partial_t + \partial_x$ (1.5)

and we also denote briefly $u_\xi = \partial_\xi u$ and $u_\eta = \partial_\eta u$. In this system (1.1),we also always assume that the initial data satisfy the compatibility condition of order one. That means $u_0(0) and\ u_1(0) = 0$ in the Dirichlet case and $u_0'(0) = 0$ and $u_1'(0) = 0$ in the Neumann case.

II.MAIN RESULT

2.1Classification of Linear Partial differential Equations of Second Order

A general linear partial differential equation of second order for a function of two independent variables x, y can be expressed as

$Ar + 2Bs + Ct + f(x, y, z, p, q) = 0$ (2.1)

where A,B,C are continuous functions of x, y defined in some domain D on the xy plane.

The classification depend on the part $Ar + 2Bs + Ct$ which is called the principle part of (1).

Definition : The linear partial differential equation (1) of the second degree defined in some region D in the xy plane is said to be

(1) Hyperbolic of a point (xy) in D, if $\Delta = B^2 - AC > 0$
(2) Parabolic , if $\Delta = B^2 - AC = 0$
(3) Elliptic , if $\Delta = B^2 - AC < 0$ at the point.

These classifications are said to hold in region Dif they hold at each point of D.
If $A, B, C$are constants then type of (2.1) is the same everywhere.
The following are typical examples of these forms:

(i)The one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$$

is hyperbolic everywhere in the xt-plane with canonical form $\frac{\partial^2 z}{\partial u \partial v} = 0.$

(ii)The one-dimensional heat (or diffusion) equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c}\frac{\partial z}{\partial t}$$

is parabolic everywhere in the xt plane.

(iii) The two-dimensional Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = 0$$

is elliptic everywhere in the xy plane.

A linear partial differential equation of second order in n independent variables $x_1, x_2, \ldots \ldots \ldots x_n$ can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad (1)$$

Where a_{ij}, b_i *and* c are constants or functions of $x_1, x_2, \ldots \ldots \ldots n$.

Let $\delta_i \equiv \frac{\partial}{\partial x_i}$ *and* $\delta_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, where $i, j = 1,2,3 \ldots \ldots \ldots n$.

Now consider $\phi = \sum_{i=n}^{n}\sum_{j=1}^{n} a_{ij}\,\delta_i \delta_j$, for all nonzero real values of δ_i.

At any point $(x_1, x_2, \ldots \ldots x_n)$ the differential equation (1) is said to be :

(i)Elliptic if ϕ is positive for all real values of δ_i and it reduces to zero only when all δ_i are zero.

(ii)Hyperbolic if ϕcan be both positive or negative and

(iii)Parabolic if the determinant

$$\Delta = \begin{vmatrix} a_{11} a_{12} \ldots \ldots a_{in}. \\ a_{21} \ \ a_{22} \ldots \ldots \ a_{2n} \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ a_{n1} \ \ a_{n2} \ldots \ldots \ldots a_{nn} \end{vmatrix} = 0$$

Example: Classify the following P.D.E and reduce to canonical form

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Solution The given P.D.E can be written as

$$F^i(u_t, u_x) = r + x^2 t \qquad (1)$$

Comparing (1) with $Ar + 2Bs + Ct + f(x, y, u, z, p, q) = 0$

We get $A = 1, B = 0, C = x^2$

$B^2 = 0, AC = 1.x^2 = x^2$.

$\therefore B^2 - AC < 0$ at all points where $x \neq 0$.

Hence the given equation is elliptic at all points except on the axis.

Also, the λ- quadratic $A\lambda^2 + 2B\lambda + C = 0$ gives

$\lambda^2 + x^2 = 0$ so that $\lambda = \pm ix$.

Hence $\frac{dy}{dx}+\lambda_1=0$ and $\frac{dy}{dx}+\lambda_2=0$becomes

$$\frac{dy}{dx}+ix=0 \;\; and \;\; \frac{dy}{dx}-ix=0.$$

Integrating these, we get

$y+\frac{1}{2}ix^2=c_1$ and $y-\frac{1}{2}ix^2=c_2$

Now to reduce (1) canconical form, we change $x,y \; to \; u,v \; and \; u,v \; to \; \alpha,\beta$ by taking

$$u=y+\frac{1}{2}ix^2=\alpha+i\beta, \quad v=y-\frac{1}{2}ix^2=\alpha-i\beta.$$

Solving for α,β, we get

$$\alpha=y \;\; and \; \beta=\frac{1}{2}x^2$$

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}.\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}.\frac{\partial \beta}{\partial x}=x\frac{\partial z}{\partial \beta}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}.\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha} so \; \frac{\partial}{\partial y}=\frac{\partial}{\partial \alpha}$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)=1.\frac{\partial z}{\partial \beta}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$=\frac{\partial z}{\partial \beta}+x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]=\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial^2 z}{\partial \alpha^2}.$$

Putting these values of $r \; and \; t$ in (1), we get

$$\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2}+x^2\frac{\partial^2 z}{\partial \alpha^2}=0$$

or

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\frac{-1}{2\beta}\frac{\partial z}{\partial \alpha}$$

canonical form of Elliptic Partial differential Equation

It satisfy the one dimension system of semi linear wave equations

$$u_{tt}-u_{xx}, \;\; t>0, x>0$$

$$u_{tt}=\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial \alpha^2}\right)=\frac{\partial^3 z}{\partial \alpha^3}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x} + x^2 \frac{\partial^2 z}{\partial \beta^2}\right) = \frac{\partial^2 z}{\partial x \partial \beta} + \frac{\partial^3 z}{\partial x \partial \beta^2}$$

$$u_{tt} - u_{xx}, \quad t > 0, x > 0$$

$$\frac{\partial^3 z}{\partial \alpha^3} - \frac{\partial^2 z}{\partial x \partial \beta} + \frac{\partial^3 z}{\partial x \partial \beta^2} \quad when\ t > 0, x > 0$$

$$B(u)t(0) = \begin{cases} u(t,0), & Dirichlet\ type \\ u_x(t,0), & Neumann\ type \end{cases}$$

$$u_{tt} - u_{xx}, \quad t > 0, x > 0$$

The Equation is also satisfy the $u(t,0),\ Dirichlet\ type$

$$r + x^2 t$$

At t=0

$$r = \frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2}$$

and $u_x(t,0)$ Neumann Boundary condition

$$= x \frac{\partial z}{\partial \beta}$$

At null coordinates are

$$\xi = \frac{t + x}{2}, \quad \eta = \frac{t - x}{2}$$

$$\xi = \frac{\frac{\partial^2 z}{\partial \alpha^2} + x}{2} \quad ,$$

$$\eta = \frac{\frac{\partial^2 z}{\partial \alpha^2} - x}{2}$$

$$u_\xi = \partial_\xi u$$

$$u_\eta = \partial_\eta u.$$

It satisfy the initial data compatibility condition of order one .

It means $u_0(0) and\ u_1(0) = 0$ in the Dirichlet case and $u_0'(0) = 0$

In Neumann case $u_0'(0) = 0$ and $u_1'(0) = 0$.

Which is the required canonical form of the given equation.

III. RESULT

This research paper satisfy the conditions of initial value problem of one-dimension semi linear Eliptic Partial Differential Equation with null condition.

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