Age-dependent shape parameter estimation from Modified Weibull Distribution

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Abstract - In the dependability literature, most generalised Weibull distributions have been proposed to suit specific data sets better than the classic two- or three-parameter Weibull model. Xie et al. introduced the modified Weibull distribution, a three-parameter Weibull distribution. This distribution is primarily linked to Chen's model, but with a scale parameter added. In this study, we attempt to estimate the age-dependent shape parameter in the vicinity of the modified Weibull distribution's additional scale parameter.

Keywords : maximum lifespan; scale parameter; shape parameter; aging; modified Weibull distribution.

I. INTRODUCTION

Gains in our understanding of age-related diseases have far outpaced advances the basic ageing mechanisms that underpin these disorders' frailty . More focus must be directed to basic research on ageing if we are to expand human life expectancy above the 15 year limit if today's primary causes of mortality were solved[4]. Best-performance life expectancy has constantly improved by a quarter of a year per year over the past 160 years, demonstrating an astonishing consistency of human progress[6].

Scientists have been looking for a better way to estimate the maximum lifespan value by creating age-related patterns at very elderly ages for a long time[2]. The reliability theory allows for the discovery of a general failure law that applies to all adult and elderly people, with the Gompertz and Weibull laws serving as particular examples of this more general failure law. [3, 7]. However, no mathematical models have been proposed to yet, including the Gompertz model, that can fully represent the growth of the death rate across the entire life span[5].

Weon et al. proposed a purely descriptive mathematical model that allows for a statistically sound estimate of the maximum human lifespan. They propose an extended Weibull model by swapping the mathematical nature of the stretched exponent as a function of age in that model[8]. They have estimated the age-dependent stretched exponent $\beta(t)$ using the mathematical expression $\beta(t) = \beta_0 + \beta_1(t) + \beta_2(t^2) + \dots$, [8].

In this article, we use τ as a function of time and represent it as $r(t)$. We were able to use the modified Weibull distribution for empirical human survival curves, particularly for women, after making this modification.

II. MODIFIED WEIBULL DISTRIBUTION

Over the last few decades, a variety of classical distributions have been widely utilised to model data in a variety of fields, including engineering, actuarial, environmental, and medical sciences, biological studies, demographics, economics, finance, and insurance. However, broader variants of these distributions are clearly needed in many applications such as lifetime analysis, finance, and insurance.

In the dependability literature, most generalised Weibull distributions have been insisted to suit specific data sets better than the classic 2 or 3 parameter model. Chen revisited a two-parameter distribution. (2000) [1]. This distribution can be bathtub-shaped or have a rising failure rate function, allowing it to fit real-life data sets. Xie et.al (2002) [9] introduced modified Weibull distribution, with the $f(t)$ defined by

$$f(t;\lambda,\tau,\alpha) = \lambda\tau\left(\frac{t}{\alpha}\right)^{\tau-1} e^{\{(\frac{t}{\alpha})^{\tau}+\lambda\alpha[1-e^{(\frac{t}{\alpha})^{\tau}}]\}}, t \geq 0,$$

where $\lambda > 0$ and $\alpha > 0$ are scale parameters and $\tau > 0$ is a shape parameter. Then survival and hazard rate are given by

$$S(t;\lambda,\tau,\alpha) = e^{\left(\lambda\alpha\left[1-e^{\left(\frac{t}{\alpha}\right)^{\tau}}\right]\right)} \tag{1}$$

and

$$h(t;\lambda,\tau,\alpha) = \mu(t) = \lambda\tau\left(\frac{t}{\alpha}\right)^{\tau-1} e^{\left(\frac{t}{\alpha}\right)^{\tau}} \tag{2}$$

The hazard function is of both bathtub shaped and an increasing function.

III. MODIFIED WEIBULL DISTRIBUTION WITH AGE - DEPENDENT SHAPE PARAMETER

If the shape parameter τ is age - dependent (represented by r(t)), then the survival function (1) takes the form

$$S(t) = e^{\left(\lambda\alpha\left[1-e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right]\right)} \tag{3}$$

Taking logarithm on both sides of (3) we get

$$lnS(t) = \lambda\alpha\left[1-e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right].$$

Further simplification gives

$$e^{\left(\frac{t}{\alpha}\right)^{r(t)}} = 1 - \frac{lnS(t)}{\lambda\alpha}$$

Taking logarithm again on both sides and simplifying further, the above equation results in

$$r(t) = \frac{ln\{ln[1-\frac{lnS(t)}{\lambda\alpha}]\}}{ln\left(\frac{t}{\alpha}\right)} \tag{4}$$

Note that when $\alpha \rightarrow \infty$, the extended Weibull distribution has the Weibull distribution as a special and asymptotic case as discussed in [9]. The mortality function $\mu(t)$ is described by the mathematical relationship with the survival function as follows

$$\mu(t) = -\frac{dlnS(t)}{dt}. \tag{5}$$

On account of (3), we get

$$\mu(t) = \frac{-d}{dt}\left[\lambda\alpha\left(1-e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right)\right]$$

$$= \lambda\alpha\frac{d}{dt}\left(e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right)$$

Therefore the mortality function for the distribution is

$$\mu(t) = \lambda\alpha e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\left(\frac{t}{\alpha}\right)^{r(t)}\left[\frac{r(t)}{t} + ln\left(\frac{t}{\alpha}\right)\frac{dr(t)}{dt}\right] \tag{6}$$

The initial concept was as follows: survival curves in their natural state exhibit (i) a quick decline in survival during the starting years of life, (ii) a somewhat stable decline, and finally (iii) an abrupt decline towards death. Surprisingly, the first behaviour matches the Weibull survival function for $r1$, while the second behaviour appears to follow the situation of $r >> 1$.

IV. ESTIMATION OF AGE - DEPENDENT SHAPE PARAMETER IN THE NEIGHBOURHOOD OF

First we observe that $r(t)$ is finite as $t \rightarrow \alpha$. For, taking limit as $t \rightarrow \alpha$ in (4) we get

$$\lim_{t\rightarrow\alpha} r(t) = \frac{ln\{ln[1-\frac{lnS(\alpha)}{\lambda\alpha}]\}}{ln\left(\frac{\alpha}{\alpha}\right)}$$

$$= \frac{ln\{ln[1-\frac{\lambda\alpha(1-e)}{\lambda\alpha}]\}}{ln\,(\frac{\alpha}{\alpha})}$$

$$= \frac{0}{0}\, form$$

Therefore by L' Hospital's rule we have

$$\lim_{t\to\alpha} r(t) = \lim_{t\to\alpha}\left\{\frac{\frac{1}{ln[1-\frac{lnS(t)}{\lambda\alpha}]}\times\frac{1}{1-\frac{lnS(t)}{\lambda\alpha}}\times(-\frac{1}{\lambda\alpha})\frac{1}{S(t)}\frac{dS(t)}{dt}}{\frac{1}{t/\alpha}\times\frac{1}{\alpha}}\right\}$$

$$= \lim_{t\to\alpha}\frac{t\mu(t)}{ln[1-\frac{lnS(t)}{\lambda\alpha}]\times[1-\frac{lnS(t)}{\lambda\alpha}]\times\lambda\alpha}$$

Hence we arrive at

$$r(\alpha) = \frac{\mu(\alpha)}{\lambda e} \tag{7}$$

Since $\mu(\alpha)$ is finite, $r(t)$ is finite as $t \to \alpha$.

Let us expand $r(t)$ in Taylor series in the neighbourhood of $t = \alpha$ upto polynomial of degree two. For, first we prove the following results using the properties of $r(t)$ and $S(t)$ from the modified Weibull model given in [9].

(i) Using (4), and simplifying further using (3) we obtain

$$\frac{dr(t)}{dt} = \frac{\mu(t)}{\lambda\alpha ln\,(t/\alpha) ln\left(1-\frac{lnS(t)}{\lambda\alpha}\right)\times\left(1-\frac{lnS(t)}{\lambda\alpha}\right)} - \frac{ln\{ln[1-\frac{lnS(t)}{\lambda\alpha}]\}}{t(ln\,(t/\alpha))^2}.$$

where $\mu(t) = -\frac{1}{S(t)}\frac{dS(t)}{dt}$. Therefore

$$ln(t/\alpha)\frac{dr(t)}{dt} = \frac{\mu(t)}{\lambda\alpha ln\left(1-\frac{lnS(t)}{\lambda\alpha}\right)\times\left(1-\frac{lnS(t)}{\lambda\alpha}\right)} - \frac{r(t)}{t}.$$

Hence

$$\lim_{t\to\alpha} ln(t/\alpha)\frac{dr(t)}{dt} = \frac{\mu(\alpha)}{\lambda\alpha e} - \frac{\mu(\alpha)}{\lambda\alpha e} \quad since \quad r(\alpha) = \frac{\mu(\alpha)}{\lambda e}$$

Thus we have

$$\lim_{t\to\alpha} ln(t/\alpha)\frac{dr(t)}{dt} = 0 \tag{8}$$

(ii) To minimize the change of the survival probability at $t = \alpha$, we must have

$$\lim_{t\to\alpha}\frac{d^2S(t)}{dt^2} = 0.$$

That is,

$$\lim_{t\to\alpha}\left[\frac{dS(t)}{dt}\mu(t) + S(t)\frac{d\mu(t)}{dt}\right] = 0,$$

since $\frac{-dS(t)}{dt} = S(t)\times\mu(t)$.

Simplifying further, we get

$$\lim_{t\to\alpha}\frac{dS(t)}{dt}\mu(\alpha) + S(\alpha)\mu'(\alpha) = 0$$

and using $S(\alpha) = e^{\lambda\alpha(1-e)}$, the last equation becomes

$$\lim_{t\to\alpha}\frac{dS(t)}{dt}\mu(\alpha) + e^{\lambda\alpha(1-e)}\mu'(\alpha) = 0. \tag{9}$$

Since

$$\mu(t) = -\frac{1}{S(t)}\frac{dS(t)}{dt},$$

$$\mu(\alpha) = \frac{-1}{S(\alpha)}\lim_{t\to\alpha}\frac{dS(t)}{dt}$$

which gives

$$\lim_{t\to\alpha} \frac{dS(t)}{dt} = -e^{\lambda\alpha(1-e)}\mu(\alpha) \tag{10}$$

Substitution of (10) into (9) gives

$$-e^{\lambda\alpha(1-e)}\mu^2(\alpha) + e^{\lambda\alpha(1-e)}\mu'(\alpha) = 0$$

from which it follows that

$$\mu'(\alpha) = \mu^2(\alpha).$$

But $\mu(\alpha) = \lambda e r(\alpha)$. Therefore

$$\mu'(\alpha) = \lambda^2 e^2 r^2(\alpha). \tag{11}$$

(iii)Now differentiation of (6) gives

$$\mu'(t) = \lambda\alpha e^{(t/\alpha)^{r(t)}}\left[(t/\alpha)^{r(t)}\left(\frac{r(t)}{t} + lnt/\alpha\frac{dr(t)}{dt}\right)\right]^2 +$$
$$\lambda\alpha e^{(t/\alpha)^{r(t)}}\left[(t/\alpha)^{r(t)}\left(\frac{tr\prime(t) - r(t)}{t^2}\right) + \frac{1}{t}\frac{dr(t)}{dt} + lnt/\alpha\frac{d^2r(t)}{dt^2}\right]$$
$$+\lambda\alpha e^{(t/\alpha)^{r(t)}}(t/\alpha)^{r(t)}\left[\frac{r(t)}{t} + lnt/\alpha\frac{dr(t)}{dt}\right]^2$$

Taking the limit as $t \to \alpha$, we obtain

$$\mu'(\alpha) = \lambda\alpha\left[\frac{2r^2(\alpha)}{\alpha^2} + \frac{2r'(\alpha)}{\alpha} - \frac{r(\alpha)}{\alpha^2}\right] + \lambda\alpha\lim_{t\to\alpha} ln\frac{t}{\alpha}r''(t) \tag{12}$$

and on account of (11), (12) becomes

$$\frac{\lambda^2 e^2 r^2(\alpha)}{\lambda\alpha} = \frac{2r^2(\alpha)}{\alpha^2} + \frac{2r'(\alpha)}{\alpha} - \frac{r(\alpha)}{\alpha^2} + \lim_{t\to\alpha} ln\frac{t}{\alpha}r''(t)$$

which gives

$$\lim_{t\to\alpha} ln\frac{t}{\alpha}r''(t) = \frac{\lambda e^2 r^2(\alpha)}{\alpha} - \frac{2r^2(\alpha)}{\alpha^2} - \frac{2r'(\alpha)}{\alpha} + \frac{r(\alpha)}{\alpha^2} \tag{13}$$

(iv) To find the limit of $\frac{dr(t)}{dt}$ as $t \to \alpha$, let us multiply and divide the term $ln(t/\alpha)$ with $dr(t)/dt$. Then as $t \to \alpha$,

$$\lim_{t\to\alpha}\frac{dr(t)}{dt} = \lim_{t\to\alpha}\frac{ln(t/\alpha)\frac{dr(t)}{dt}}{ln(t/\alpha)} = \left(\frac{0}{0}\right) \; form$$

Hence application of L'Hospital's rule results in

$$\lim_{t\to\alpha}\frac{dr(t)}{dt} = \lim_{t\to\alpha}\left[\frac{1/tr\prime(t) + ln(t/\alpha)\frac{d^2r(t)}{dt^2}}{1/t}\right]$$

$$r'(\alpha) = r'(\alpha) + \lim_{t\to\alpha} ln(t/\alpha)\frac{r''(t)}{1/t}$$

Using (13) and simplifying we obtain

$$r'(\alpha) = \frac{\lambda e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} \tag{14}$$

(v) The following property is useful to determine the limit of $\frac{d^2r(t)}{dt^2}$ when $t \to \alpha$. To minimize the change of the survival probability, it is well known that $\frac{d^2}{dt^2}\left[\frac{-dS(t)}{dt}\right] < 0$ which in turn implies that $\frac{d^2}{dt^2}(\mu S) < 0$, since $\mu(t) = \frac{-1}{S(t)}\frac{dS(t)}{dt}$. It is clear that

$$\lim_{t\to\alpha}\frac{d^2}{dt^2}(\mu S) = \mu''(\alpha)S(\alpha) + 2\mu'(\alpha)S'(\alpha)$$
$$= \mu''(\alpha)S(\alpha) - 2\mu^3(\alpha)S(\alpha)$$

since $\frac{d^2S}{dt^2}/_{t=\alpha} = 0$ and $S'(\alpha) = -\mu(\alpha)S(\alpha)$. Therefore $\lim_{t\to\alpha}\frac{d^2}{dt^2}(\mu S) < 0$ implies that

$$\mu''(\alpha) < 2\mu^3(\alpha). \tag{15}$$

(vi) Differentiating (6) twice in t , and as t tends to α we get

$$\mu''(\alpha) = \lambda\alpha\left[\frac{2r^3(\alpha)}{\alpha^3}\left(1 + \frac{2r'(\alpha)}{\alpha} - \frac{r(\alpha)}{\alpha}\right) + \frac{r^2(\alpha)}{\alpha^2} + \frac{3r(\alpha)}{\alpha}\left(\frac{2r'(\alpha)}{\alpha} - \frac{r(\alpha)}{\alpha^2}\right)\right]$$

$$+\lambda\alpha\left[\frac{3r''(\alpha)}{\alpha}-\frac{3r'(\alpha)}{\alpha^2}+\frac{2r(\alpha)}{\alpha^3}\right]+\lambda\alpha\lim_{t\to\alpha} ln(t/\alpha)r'''(t).$$

Combining (15) and the above equation we get

$$\lim_{t\to\alpha} ln(t/\alpha)r'''(t) < \frac{2\mu^3(\alpha)}{\lambda\alpha}-\left[\frac{2r^3(\alpha)}{\alpha^3}\left(1+\frac{2r'(\alpha)}{\alpha}-\frac{r(\alpha)}{\alpha}\right)+\frac{r^2(\alpha)}{\alpha^3}\right]$$
$$-\left[\frac{3r(\alpha)}{\alpha}\left(\frac{2r'(\alpha)}{\alpha}-\frac{r(\alpha)}{\alpha^2}\right)+\frac{3r''(\alpha)}{\alpha}-\frac{3r'(\alpha)}{\alpha^2}+\frac{2r(\alpha)}{\alpha^3}\right].$$

Using (7) for $r(\alpha)$ and (14) for $r'(\alpha)$ we obtain

$$\lim_{t\to\alpha} ln(t/\alpha)r'''(t) < \frac{2\lambda^2e^3r^3(\alpha)}{\alpha}-\frac{2r^3(\alpha)}{\alpha^3}-\frac{3r''(\alpha)}{\alpha}-\frac{2r(\alpha)}{\alpha^3}$$

(vii) To find the limit of $\frac{d^2r(t)}{dt^2}$ as $t\to\alpha$, let us multiply and divide the term $ln(t/\alpha)$ with $\frac{d^2r(t)}{dt^2}$. Then

$$\lim_{t\to\alpha} r''(t)=\lim_{t\to\alpha}\frac{ln(t/\alpha)r''(t)}{ln(t/\alpha)}$$

as $t\to\alpha$, $d^2r(t)/dt^2$ takes ($\frac{0}{0}$ form). Hence applying L'Hospital's rule to the RHS , we get

$$\lim_{t\to\alpha} r''(t)=\lim_{t\to\alpha}\left[\frac{ln(t/\alpha)r'''(t)+\frac{r''(t)}{t}}{\frac{1}{t}}\right]$$

and using (??) the last equation becomes

$$r''(\alpha) < r''(\alpha)+\alpha\left[\frac{2\lambda^2e^3r^3(\alpha)}{\alpha}-\frac{2r^3(\alpha)}{\alpha^3}-\frac{3r''(\alpha)}{\alpha}-\frac{2r(\alpha)}{\alpha^3}\right]$$
$$=-2r''(\alpha)+2\lambda^2e^3r^3(\alpha)-\frac{2r^3(\alpha)}{\alpha^2}-\frac{2r(\alpha)}{\alpha^2}$$

Hence we have

$$r''(\alpha) < \frac{2\lambda^2e^3r^3(\alpha)}{3}-\frac{2r^3(\alpha)}{3\alpha^2}-\frac{2r(\alpha)}{3\alpha^2}$$

That is,

$$r''(\alpha)=\frac{2\lambda^2e^3r^3(\alpha)}{3}-\frac{2r^3(\alpha)}{3\alpha^2}-\frac{2r(\alpha)}{3\alpha^2}-k \quad (16)$$

where k is an unknown constant and $k>0$. Using (14) and (16), the expansion of $r(t)$ in Taylor series in the neighbourhood of $t=\alpha$ takes the form

$$r(t)=r(\alpha)+\frac{\lambda e^2r^2(\alpha)}{2}-\frac{r^2(\alpha)}{\alpha}+\frac{r(\alpha)}{2\alpha}(t-\alpha)+\frac{\frac{2\lambda^2e^3r^3(\alpha)}{3}-\frac{2r^3(\alpha)}{3\alpha^2}-\frac{2r(\alpha)}{3\alpha^2}-k}{2!}(t-\alpha)^2. \quad (17)$$

To determine k, let us make use of the following property. $S(t)$ is mathematically a monotonic decay function of age ($\frac{dS(t)}{dt}<0$) between $S(0)=1$ and $S(t_m)=0$. If the change in survival probability is minimal, the preceding equation indicates a mathematical tendency ($\frac{dS(t)}{dt}\to 0$), then at young ages, the slope of the stretched exponent falls below a particular positive value ($t<\alpha$), while at older ages, it is greater than a specific negative value $(t>\alpha)$. The mathematical limit of each survival curve's maximum human lifespan (tm) could be estimated at

$$\frac{dr(t)}{dt}=\frac{-r(t)}{t\ ln(t/\alpha)} \quad (18)$$

and as $t\to t_m$ we get

$$\frac{dr}{dt}/_{t=t_m}=-\frac{r(t_m)}{t_m\ ln(t_m/\alpha)}. \quad (19)$$

From (17) we have

$$\frac{dr(t)}{dt}/_{t=t_m}=\frac{\lambda e^2r^2(\alpha)}{2}-\frac{r^2(\alpha)}{\alpha}+\frac{r(\alpha)}{2\alpha}$$

$$+\left[\frac{2\lambda^2 e^3 r^3(\alpha)}{3}-\frac{2r^3(\alpha)}{3\alpha^2}-\frac{2r(\alpha)}{3\alpha^2}-k\right](t_m-\alpha).$$

Equating (19) and the above equation we get

$$\frac{-r(t_m)}{t_m\ ln(t_m/\alpha)}=\frac{\lambda e^2 r^2(\alpha)}{2}-\frac{r^2(\alpha)}{\alpha}+\frac{r(\alpha)}{2\alpha}$$
$$+\left[\frac{2\lambda^2 e^3 r^3(\alpha)}{3}-\frac{2r^3(\alpha)}{3\alpha^2}-\frac{2r(\alpha)}{3\alpha^2}-k\right](t_m-\alpha).$$

Solving for k results in

$$k=\frac{r(t_m)}{t_m(t_m-\alpha)\ ln(t_m/\alpha)}+\left[\frac{2\lambda^2 e^3 r^3(\alpha)}{3}-\frac{2r^3(\alpha)}{3\alpha^2}-\frac{2r(\alpha)}{3\alpha^2}\right]$$
$$+\frac{\lambda e^2 r^2(\alpha)}{2(t_m-\alpha)}-\frac{r^2(\alpha)}{\alpha(t_m-\alpha)}+\frac{r(\alpha)}{2\alpha(t_m-\alpha)}.$$

Substitution of k into (17) finally gives

$$r(t)=r(\alpha)+\left[\frac{\lambda e^2 r^2(\alpha)}{2}-\frac{r^2(\alpha)}{\alpha}+\frac{r(\alpha)}{2\alpha}\right](t-\alpha)$$
$$-\left[\frac{r(t_m)}{t_m(t_m-\alpha)\ ln(t_m/\alpha)}+\frac{\lambda e^2 r^2(\alpha)}{2(t_m-\alpha)}-\frac{r^2(\alpha)}{\alpha(t_m-\alpha)}+\frac{r(\alpha)}{2\alpha(t_m-\alpha)}\right]\frac{(t-\alpha)^2}{2!}.$$

The above equation gives the required estimation for the shape parameter $r(t)$ in the neighbourhood of the scale parameter α.

V. CONCLUSION

The objective of estimating the shape parameter in the neighbourhood of α is to provide an estimation for the maximum lifespan especially in women. This maximum lifespan prediction may enable us to prove some interesting aging related problems in women. The impact of another scale parameter λ is not addressed here. Future studies of the shape parameter could improve our understanding of aging in women.

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