Domination Number To the Kronecker Product of Some Connected Graphs

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Abstract : Let G_1 and G_2 be two graphs.The Kronecker product G_1 (K) G_2 has vertex set V(G_1 (K) G_2)= V (G_1) × V (G_2) and edge set E (G_1 (K) G_2) = $\{(u_1, v_1)\ (u_2, v_2)/u_1u_2 \in E(G_1)\ , v_1v_2 \in E(G_2)\}$. In this paper, we have found the domination number and the dominating sets to Kronecker product of $K_{1,n}$ with its transformation graphs. Also we have discussed some results with wheel graph..

Keywords : Kronecker product, domination number, dominating set, wheel graph, star graph.

I. INTRODUCTION

Domination is an interesting research area in graph theory. Wu and Meng have studied the concept of graph transformation and many applications have been studied in this topic.A graph G consists of a pair (V(G),E(G)) where V(G) is a non empty finite set whose elements are called vertices and E(G) is a set of unordered pairs of distinct elements of V(G). A graph that contains no cycles is called an acyclic graph. A connected acyclic graph is called a tree. $K_{1,n}$ is called the star Graph.

For S ⊆ V, if every vertex of V is either an element of S or V-S is said to be a dominating set and the corresponding dominating set is called a γ-set of G. The open neighborhood $N(v)$ of $v \in V$ is the set of vertices adjacent to v, that is, $N(v) = \{u/uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A wheel graph is a cycle of length at least 3, plus a single point in the center connected by "spokes" to every point on the cycle.

Let $G = (V(G), E(G))$ be a graph and x, y,z be three variables taking values + or -. The **transformation graph G^{xyz}** is the graph having $V(G) \cup E(G)$ as the vertex set and for $\propto, \beta \in V(G) \cup E(G)$, $\propto and\ \beta$ are adjacent in G^{xyz} if and only if one of the following holds:

(i) $\propto, \beta \in V(G)$. $\propto and\ \beta$ are adjacent in G if x = +; $\propto and\ \beta$ are not adjacent in G if x = - .

(ii) $\propto, \beta \in E(G)$. $\propto and\ \beta$ are adjacent in G if y = +; $\propto and\ \beta$ are not adjacent in G if y = - .

(iii) $\propto \in V(G), \beta \in E(G). \propto$ *and* β are incidentt in G if z = +; $\propto$ *and* β are not incident in G if z = - .

II. MAIN RESULTS

Theorem : 1

Let $G^* = G\ (K)\ G^{-+-}$ where $G = K_{1,n}$ then $\gamma(G^*) = 5$ and $|D(G^*)| = n^2(2n-1)$.

Proof:

In G, $d(u_0) = n$ and $d(u_i) = 1$ for $1 \le i \le n$

In G^{-+-}, $d(v_0) = 0,\ d(v_i) = d(e_i) = 2(n-1)$ for all $i = 1,2,\dots,n$.

In $G^* = G\ (K)\ G^{-+-}$, $d(u_0v_0) = 0$ and $d(u_0v_i) = d(v_0e_i) = 2(n-1), 1 \le i \le n$.

$N(u_0v_i) = \{u_jv_k / 1 \le j,k \le n, j \ne 0\ \&\ i \ne k\}$

$\{u_jv_i, u_je_i / 1 \le i \le n\} \notin N(u_0v_i)$ for all i.

$\{u_jv_i, u_je_i\} \in N(u_0v_k)$, $k \ne i$.

Arrange the elements of G^* is of the matrix form

$$\begin{pmatrix} u_0v_1 & u_0v_2 & u_0v_3 & \dots & u_0v_n & u_oe_1 & u_0e_2 & u_0e_3 & \dots & u_0e_n \\ u_1v_1 & u_1v_2 & u_1v_3 & \dots & u_1v_n & u_1e_1 & u_1e_2 & u_1e_3 & \dots & u_1e_n \\ u_2v_1 & u_2v_2 & u_2v_3 & \dots & u_2v_n & u_2e_1 & u_2e_2 & u_2e_3 & \dots & u_2e_n \\ u_3v_1 & u_3v_2 & u_3v_3 & \dots & u_3v_n & u_3e_1 & u_3e_2 & u_3e_3 & \dots & u_3e_n \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ u_nv_1 & u_nv_2 & u_nv_3 & \dots & u_nv_n & u_ne_1 & u_ne_2 & u_ne_3 & \dots & u_ne_n \end{pmatrix}$$

Choose any two elements from the first row of the form $\{u_0v_i, u_0v_j / u_0v_i, u_0e_j / u_0e_i, u_0e_j\}$ such that $i \ne j$ and $1 \le i,j \le n$ then another two elements which lies on the same column which was already selected which is of the form $\{u_kv_i, u_kv_j / u_kv_i, u_ke_j / u_ke_i, u_ke_j\}$ such that $k \ne 0$ then

$N\{u_0v_i, u_0v_j, u_kv_i, u_kv_j\} = N(u_0v_i, u_0e_j, u_kv_i, u_ke_j\}$

$$= N(u_0 e_i, u_0 e_j, u_k e_i, u_k e_j)$$

$$= V(G) - \{u_0 v_0\} \qquad \ldots\ldots. \ (1)$$

Since $\{u_0 v_0\}$ is an isolated vertex of G^* the set containing four elements represented in (1) together with $\{u_0 v_0\}$ is the required dominating set of G^*.

Hence, $\gamma(G^*) = 5$

We choose 2 elements from the first row containing $2n$ elements in $(2n)C_2$ ways and the remaining 2 elements from the n rows can be selected in n ways.

Hence $|D(G^*)| = n[(2n)C_2] \ = n^2(2n-1)$

Theorem : 2

Let $G^* = G\ (K)\ G^{+--}$ where $G = K_{1,n}$ then $\gamma(G^*) = 4$.

Proof:

Let $V(G) = \{u_i / \ 0 \le i \le n\}$ with $d(u_0) = n$ and $d(u_i) = 1$ for all $i = 1,2, \ldots, n$.

$V(G^{-++}) = \{v_i, e_j / \ 0 \le i \le n; 1 \le j \le n\}$ with $d(v_i) = n$ for all $0 \le i \le n$; $d(e_i) = n-1$ for all $i = 1,2,3, \ldots, n$.

Let $V(G^*) = \{u_i v_j, u_i e_k / \ 0 \le i, j \le n; 1 \le k \le n\}$

with $d(u_0 v_j) = n^2;\ 0 \le j \le n$

$d(u_i v_j) = n, 1 \le i, j \le n$

$d(u_i v_0) = n\ ; 1 \le i \le n$

$d(u_i e_j) = n-1\ ; 1 \le i, j \le n$

$d(u_0 e_j) = n(n-1)\ ; 1 \le j \le n$

$N(u_0v_0) = \{u_iv_j / 1 \le i,j \le n\}$

$N(u_0v_i) = \{u_jv_0 / 1 \le j \le n\}$ for all i, $\{u_je_k / k \ne i, 1 \le j,k \le n\}$

Also $\{u_0v_j, u_0e_j / 1 \le j \le n\} \subseteq N(u_iv_i, u_iv_k) / j \ne k$ for all $i = 1,2,\dots,n$.

Hence, any four elements of the form

$\{u_0v_0, u_0v_i, u_jv_k, u_jv_l / 1 \le j,k,l \le n \,\&\, k \ne l\}$ for all i dominates the elements of G^*.

$\Rightarrow \gamma(G^*) = 4$.

Theorem : 3

$G^* = G\ (K)\ G^{---}$, then $\gamma(G^*) = 5$.

Proof:

Let $V(G) = \{u_i / 0 \le i \le n\}$ with $d(u_0) = n$ and $d(u_i) = 1$ for all $i = 1,2,\dots,n$.

$E(G) = \{e_i = u_0u_i, 0 \le i \le n\}$ such that $e_i = u_0u_i, 1 \le i \le n$.

$V(G^{---}) = \{v_i, e_j / \ 0 \le i \le n; 1 \le j \le n\}$ such that $d(v_i) = 0$, $d(v_i) = 2n - 1$ for all i $d(e_j) = n - 1, 1 \le j \le n$.

Let $G^* = G\ (K)\ G^{---}$ and $V(G^*) = \{u_iv_j, u_ie_k / 0 \le i,j \le n; 1 \le k \le n\}$ with $d(u_0v_0) = 0$

$d(u_0v_j) = n(2n-1); d(u_iv_j) = 2n - 1, 1 \le i,j \le n$.

$N(u_0v_i) = \{u_iv_j, u_ie_j / 1 \le i,j \le n \,\&\, i \ne j\}$

$N(u_iv_k) = \{u_ov_j, u_oe_j / 1 \le j \le n, j \ne k\}$ for all $i = 1,2,\dots,n$

Hence, any set consists four elements of the form $\{u_0v_i, u_ov_j / i \ne j\} \cup \{u_iv_j, u_kv_l / j \ne l, 1 \le i,j,k,l \le n\}$ dominates all the elements of G^* other than $\{u_ov_0\}$.

Also $\{u_ov_0\}$ is a $\gamma -$ required vertex of G^*. Hence, $\gamma(G^*) = 5$.

Result : 1

$G = K_{1,n}$, G^T is any transformation of G and $G^* = G\ (K)\ G^T$ then $\gamma(G^*) \leq 5$.

Theorem : 4

For any simple connected graph G_1 and G_2 , $\gamma[G_1(K)G_2] > 1$

Proof:

Let $(G_1) = \{u_i / 1 \leq i \leq m\}$, $V(G_2) = \{v_j / 1 \leq j \leq n\}$

$$V(G) = \{u_i v_j / u_i \in G_1 \text{ and } v_j \in G_2\}$$

Since G_1 and G_2 are simple,

$$d(u_i) \leq m - 1 \text{ for all } i = 1,2,\dots,m$$

$$d(v_j) \leq n - 1 \text{ for all } j = 1,2,\dots,n$$

$$d(u_i v_j) \leq (m-1)(n-1) \text{ for all } u_i v_j \in V(G)$$

$$d(u_i v_j) \leq mn - m - n - 1$$

$$d(u_i v_j) \leq mn - (m+n) \text{ for all } i, j.$$

Hence, no vertices of G is of degree $mn - 1$.

That is all vertices of $V(G)$ cannot be dominated by a simple vertex of $V(G)$.

$$\Rightarrow \gamma(G) > 1.$$

Result : 2

$G = W_{1,n}$ then $\gamma(G) = 1$

Proof:

Let $G = W_{1,n}$, Let $V(G) = \{v_i / 0 \le i \le n\}$ with $d(v_0) = n$ and $d(v_i) = 1$ for all $i = 1,2, \dots, n$.

$$N(v_i) = \{v_0, v_{i-1}, v_{i+1} / 2 \le i \le n-1\}, N(v_1) = \{v_2, v_n, v_0\} : N(v_n) = \{v_0, v_1, v_{n-1}\}$$

Clearly, $N(v_0) = V(G)$, hence $D(G) = \{v_0\} \Rightarrow \gamma(G) = |D(G)| = 1$.

Result : 3

$G = W_{1,n}$ then $\gamma(\bar{G}) = 3$ and $|D(G)| = nC_2 - n$.

Proof:

By result : 6, for all element $v_i \in V(\bar{G}), 0 \le i \le n$

$d(v_0) = 0$ and $d(v_i) = n - 3, 1 \le i \le n$

Since $v_i v_{i+1}, v_1 v_n \notin E(\bar{G}), 1 \le i \le n-1$

$D = \{(v_i, v_j) \cup \{v_0\} / |i - j| = 2, n-2\}$ are the dominating sets of G with cardinality 3.

$$\Rightarrow \gamma(\bar{G}) = 3 \text{ and } |D(G)| = nC_2 - n.$$

Theorem : 5

Let $G = W_{1,n}$ be any wheel graph with n+1 vertices, $G^* = G\ (K)\ \bar{G}$ then $\gamma(G^*) = n + 5$.

Proof:

Let $V(G) = \{u_i / 0 \le i \le n\}$ with $d(u_0) = n$, $d(u_i) = 3, 1 \le i \le n$

$$N(u_i) = \{u_0, u_{i-1}, u_{i+1} / 2 \le i \le n-1\}$$

$$N(v_1) = \{u_0, u_2, u_n\} : N(u_n) = \{u_0, u_1, u_{n-1}\}$$

$V(\bar{G}) = \{v_i / 0 \le i \le n\}$ with $d(v_0) = 0$ and $d(v_i) = n - 3, 1 \le i \le n$

$$G^* = G\ (K)\ \bar{G}$$

$$V(G^*) = \{u_i v_j / 0 \le i \le n\}, d(u_i v_0) = 0, 0 \le i \le n$$

Let $S_1 = \{u_0v_i, u_0v_j / |i-j| \geq 3\}$

$S_2 = \{u_kv_i, u_lv_j / k, l = 1,2,\dots,n\}$

$D_1 = \{u_iv_0 / 0 \leq i \leq n\}$

Any pair of vertices $(u_0v_i, u_0v_j) \in S_1$, $(u_kv_i, u_lv_j) \in S_2$

$N(u_0v_i, u_0v_j) \cup N(u_kv_i, u_lv_j) \cup D_1 = V(G^*)$ for all i, j, k, l.

Hence, $\gamma(G^*) = n + 5$.

Result : 4

$G = W_{1,n}$, then $\gamma(G^*) > \gamma(G)(K)\gamma(\bar{G})$

Theorem: 6

Let$G_1 = K_{1,m}$, $G_2 = K_n$, $G = G_1(K)G_2$ then $\gamma(G) = 3$ and $|D(G)| = 2m(nC_2)$

Proof:

Given$G_1 = K_{1,m}$, $G_2 = K_n$

Let $V(G_1) = \{u_i / 1 \leq i \leq m\}$ with $d(u_0) = m$, $d(u_i) = 3, 1 \leq i \leq m$

$V(G_2) = \{v_i / 1 \leq i \leq n\}$ & $d(v_i) = n-1, 1 \leq i \leq n$

$V(G) = \{u_iv_j / 0 \leq i \leq m, 1 \leq j \leq n\}$ with $d(u_0v_i) = n(n-1)$ for all $i = 1,2,\dots,n$

$d(u_iv_j) = 3(n-1), 1 \leq i \leq m, 1 \leq j \leq n.$

Let $S_1 = \{u_0v_i / 1 \leq i \leq n\}$

$S_1^* = \{(u_0v_i, u_0v_j) / 1 \leq i, j \leq n \ \& \ i \neq j\}$

$S_2 = \{u_kv_i / u_kv_j / 1 \leq k \leq m\}$

Now any two elements from S_1^* together with an elements of S_2, union of its neighbours forms the vertex set of $V(G)$.

[For example,

$$N(u_0v_1) \cup N(u_0v_3) \cup N(u_3v_1) = V(G)$$

$$N(u_0v_1) \cup N(u_0v_3) \cup N(u_5v_3) = V(G)]$$

Which is the required minimum dominating set of G.

$$\Longrightarrow \gamma(G) = 3$$

$|S_1| = n$ & $|S_2| = 2m$ and hence, $|D(G)| = 2m(nC_2)$.

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