q& (p,q)-Derivative of Gaussian function: f(x)= e^{-ax^2}, a>0

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Abstract:In this paper the authors have derived q & (p,q)- Derivative of special function i.e Gaussian function f(x)= e^{-ax^2}, a>0 with some special cases.

Keywords: q-derivative, (p,q)-derivative, Gaussian function, exponential function, basic numbers $[n]_p$ and $[n]_{p,q}$.-

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I.INTRODUCTION

The quantum calculus is an ordinary calculus without taking the limit. More precisely, in the classic calculus the derivative of a function f(x) is defined as:

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

subject to the existence of the limit. The q -derivative is defined by:

$$\frac{f(qx) - f(x)}{(q-1)x}$$

where q is a fixed scalar different from 1. Note that this type of derivatives does not use the limit.

This is a special type of quantum calculus. Quantum derivative can be defined in many different ways. For instance, we can define different types of quantum derivatives f'(x), where we do not take the limit. By considering y = qx, y = x + h and y = x^p, the derivatives are then called q-derivative, h-derivative, and p -derivative respectively of the f(x) where p,q are a fixed number different from 1 and h is a fixed number different from 0.

These quantum derivatives (q-derivative, h-derivative and p-derivative) for the function f(x) = x^3 are respectively:

$$\frac{(qx)^3 - x^3}{(q-1)x} = (q^2 + q + 1)x^2$$

$$\frac{(x+h)^3 - x^3}{h} = 3x^2 + 3hx + h^2$$

and $$\frac{x^{3p}-x^3}{x^p-x}=\frac{x^{3(p-1)}-1}{x^{p-1}-1}x^2$$

The history of q -calculus goes back to Euler and Jacobi in the eighteenth century and F.H. Jackson, in the early twentieth century, revisited their related work. In the second half of the twentieth century, a remarkable research interest in the area of the q -calculus was observed due to its applications in several areas of mathematics and physics. Recently, a tremendous interest was driven by high demands for mathematical models in quantum computing. The q-calculus bridges a connection between mathematics and physics. It has a lot of applications in research areas such as number theory, combinatorics, orthogonal polynomials, hyper geometric functions, quantum theory, and electronics.

For more detailed material on quantum calculus, see the monographs [2] [3] [4] and [5] and references therein. Following the notations and terminology from the references [1] [6] [7]and [8] for an arbitrary function f(x) the q-differential is defined by:

$$(d_q f)(x)=f(qx)-f(x)$$

In particular, let

$$d_q x = (q-1)x,$$

then the definition of the q-derivative in the sense of Jackson is as follows:

$$D_q\ f(x)=\frac{d_q}{d_q\ x}f(x)=\frac{f(qx)-f(x)}{(q-1)x} \text{ where } x\neq 0,\ 0<q<1 \qquad \text{(eq. 1)}$$

And for $x = 0$,

$$\frac{d_q}{d_q\ x}f(0)=\lim_{x\to 0}\frac{d_q f(x)}{d_q x}$$

Clearly, if f(x) is differentiable, then

$$\lim_{q\to 1}\left(\frac{d_q}{d_q\ x}f\right)(x)=\frac{df(x)}{dx}$$

For any real number n, let us define

$$[n]_q=\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-1}$$

Such that

$$q^n-1=[n]_q\ (q-1)$$

Or $1-q^n=[n]_q\ (1-q)$ (eq. 2)

We now review briefly some concepts of post quantum calculus i.e. (p,q) calculus. The (p,q) derivative of a function f with respect to x is defined as

$$D_{p,q}\ f(x) = \frac{d_{p,q}}{d_{p,q}\ x} f(x) = \frac{f(px) - f(qx)}{(p-q)x}\ , x \neq 0 \qquad \text{(eq. 3)}$$

And

$$\left(D_{p,q}\ f\right)(0) = f'(0), \text{ provided that f is differentiable at 0.}$$

For any number n, twin number or (p,q) number is defined as:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + \ldots + pq^{n-2} + q^{n-1}$$

Such that

$$p^n - q^n = [n]_{p,q}\ (p-q) \quad \text{(eq. 4)}$$

Which is a natural generalization of the q-number such that:

$$[n]_{p,q}\ |_{\ p=1} = [n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}$$

Note that

$$[n]_{p,q} = [n]_{q,p}$$

Taking here p=1, all the notations given in this part reduces to the notation of the usual q-calculus[1][6] [8].

II. MAIN RESULTS

Gaussian functions are widely used in statistics to describe the normal distributions, in signal processing to define Gaussian filters, in image processing where two-dimensional Gaussians are used for Gaussian blurs, and in mathematics to solve heat equations and diffusion equations and to define the Weierstrass transform. Here we will find the q-derivative and (p,q) derivative of the following form of Gaussian function: f(x)= e^{-ax^2}, a>0

III. RESULT FIRST

Let f(x) = e^{-ax^2}, a>0 be a Gaussian function.

Then $(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}$ [by using eq.1]

$$= \frac{e^{-a(qx)^2} - e^{-ax^2}}{(q-1)x}$$

$$= \frac{1}{(q-1)x}\left[e^{-a(qx)^2} - e^{-ax^2}\right]$$

$$= \frac{1}{(q-1)x}\left[1 - \frac{a(qx)^{2}}{1!} + \frac{[a(qx)^2]^{2}}{2!} - \frac{[a(qx)^2]^{3}}{3!} + \frac{[a(qx)^2]^{4}}{4!} - \cdots \ldots - \left\{1 - \frac{ax^2}{1!} + \frac{(ax^2)^{2}}{2!} - \frac{(ax^2)^{3}}{3!} + \frac{(ax^2)^{4}}{4!} - \cdots \ldots\ldots\right\}\right]$$

$$= \frac{1}{(q-1)x}\left[ax^2 - a(qx)^{2} + \frac{[a(qx)^2]^{2}}{2!} - \frac{(ax^2)^{2}}{2!} + \frac{(ax^2)^{3}}{3!} - \frac{[a(qx)^2]^{3}}{3!} + \frac{[a(qx)^2]^{4}}{4!} - \frac{(ax^2)^{4}}{4!} + \cdots\right]$$

$$= \frac{1}{(q-1)x}[ax^2(1-q^2) + \frac{(ax^2)^{2}}{2!}(q^4-1) + \frac{(ax^2)^{3}}{3!}(1-q^6) + \frac{(ax^2)^{4}}{4!}(q^8-1) + \cdots]$$

$$= \frac{1}{(q-1)x}\left[ax^2(1-q^2) - \frac{(ax^2)^{2}}{2!}(1-q^4) + \frac{(ax^2)^{3}}{3!}(1-q^6) - \frac{(ax^2)^{4}}{4!}(1-q^8) + \cdots\right]$$

$$= \frac{1}{(q-1)x}\left[ax^2[2]_q\,(1-q) - \frac{(ax^2)^{2}}{2!}[4]_q\,(1-q) + \frac{(ax^2)^{3}}{3!}[6]_q\,(1-q) - \frac{(ax^2)^{4}}{4!}[8]_q\,(1-q) + \cdots\right]$$

[Using eq. 2]

$$= \frac{1}{(q-1)x}(1-q)ax^2\left[[2]_q - \frac{ax^2}{2!}[4]_q + \frac{(ax^2)^{2}}{3!}[6]_q - \frac{(ax^2)^{3}}{4!}[8]_q + \cdots\right]$$

$$= -ax\left[[2]_q - \frac{ax^2}{2!}[4]_q + \frac{(ax^2)^{2}}{3!}[6]_q - \frac{(ax^2)^{3}}{4!}[8]_q + \cdots\right] \text{(eq. 5)}$$

IV. RESULT SECOND

Let f(x) = e^{-ax^2}, a>0 be a Gaussian function.

Then $(D_{q,p}f)(x) = \frac{f(qx)-f(px)}{(q-p)x}$ [by using eq. 3]

$$= \frac{e^{-a(qx)^2} - e^{-a(px)^2}}{(q-p)x}$$

$$= \frac{1}{(q-p)x}\left[e^{-a(qx)^2} - e^{-a(px)^2}\right]$$

$$=\frac{1}{(q-p)x}\left[1-\frac{a(qx)^{2}}{1!}+\frac{[a(qx)^{2}]^{2}}{2!}-\frac{[a(qx)^{2}]^{3}}{3!}+\frac{[a(qx)^{2}]^{4}}{4!}-\cdots\ldots-\left\{1-\frac{a(px)^{2}}{1!}+\frac{[a(px)^{2}]^{2}}{2!}-\frac{[a(px)^{2}]^{3}}{3!}+\frac{[a(px)^{2}]^{4}}{4!}-\cdots\ldots\right\}\right]$$

$$=\frac{1}{(q-p)x}\left[a(px)^{2}-a(qx)^{2}+\frac{[a(qx)^{2}]^{2}}{2!}-\frac{[a(px)^{2}]^{2}}{2!}+\frac{[a(px)^{2}]^{3}}{3!}-\frac{[a(qx)^{2}]^{3}}{3!}+\frac{[a(qx)^{2}]^{4}}{4!}-\frac{[a(px)^{2}]^{4}}{4!}+\cdots\right]$$

$$=\frac{1}{(q-p)x}\left[ax^{2}(p^{2}-q^{2})+\frac{(ax^{2})^{2}}{2!}(q^{4}-p^{4})+\frac{(ax^{2})^{3}}{3!}(p^{6}-q^{6})+\frac{(ax^{2})^{4}}{4!}(q^{8}-p^{8})+\cdots\right]$$

$$=\frac{1}{(q-p)x}\left[ax^{2}(p^{2}-q^{2})-\frac{(ax^{2})^{2}}{2!}(p^{4}-q^{4})+\frac{(ax^{2})^{3}}{3!}(p^{6}-q^{6})-\frac{(ax^{2})^{4}}{4!}(p^{8}-q^{8})+\cdots\right]$$

$$=\frac{1}{(q-p)x}\left[ax^{2}[2]_{q,p}(p-q)-\frac{(ax^{2})^{2}}{2!}[4]_{q,p}(p-q)+\frac{(ax^{2})^{3}}{3!}[6]_{q,p}(p-q)-\frac{(ax^{2})^{4}}{4!}[8]_{q,p}(p-q)+\cdots\right]$$

[Using eq. 4]

$$=\frac{1}{(q-p)x}(p-q)ax^{2}\left[[2]_{q,p}-\frac{ax^{2}}{2!}[4]_{q,p}+\frac{(ax^{2})^{2}}{3!}[6]_{q,p}-\frac{(ax^{2})^{3}}{4!}[8]_{q,p}+\cdots\right]$$

$$=-ax\left[[2]_{q,p}-\frac{ax^{2}}{2!}[4]_{q,p}+\frac{(ax^{2})^{2}}{3!}[6]_{q,p}-\frac{(ax^{2})^{3}}{4!}[8]_{q,p}+\cdots\right]\text{(eq. 6)}$$

V. SPECIAL CASES

[A]. Take q=1 in the equation 5,we will get the classicalderivative of given Gaussian function.

$$\frac{df}{dx}=-ax\left[[2]_{1}-\frac{ax^{2}}{2!}[4]_{1}+\frac{(ax^{2})^{2}}{3!}[6]_{1}-\frac{(ax^{2})^{3}}{4!}[8]_{1}+\cdots\right]$$

$$=-ax\left[2-\frac{4ax^{2}}{2!}+\frac{6(ax^{2})^{2}}{3!}-\frac{8(ax^{2})^{3}}{4!}+\cdots\right]$$

$$=-ax.2\left[1-\frac{2ax^{2}}{2!}+\frac{3(ax^{2})^{2}}{3!}-\frac{4(ax^{2})^{3}}{4!}+\cdots\right]$$

$$=-2ax\left[1-\frac{ax^{2}}{1!}+\frac{(ax^{2})^{2}}{2!}-\frac{(ax^{2})^{3}}{3!}+\cdots\right]$$

$$=-2ax.e^{-ax^{2}}$$

[B]. Take p=1 in the equation 6, we will get the q-derivative of given Gaussian function.

$$(D_{q,1}\, f)(x) = -ax\left[[2]_{q,1} - \frac{ax^2}{2!}[4]_{q,1} + \frac{(ax^2)^2}{3!}[6]_{q,1} - \frac{(ax^2)^3}{4!}[8]_{q,1} + \cdots\right]$$

$$(D_{q}\, f)(x) = -ax\left[[2]_{q} - \frac{ax^2}{2!}[4]_{q} + \frac{(ax^2)^2}{3!}[6]_{q} - \frac{(ax^2)^3}{4!}[8]_{q} + \cdots\right]$$

VI. CONCLUSION

The results proved in this paper give some contribution to the p, (p,q)-derivatives and are believed to be new and fruitful for quantum and post quantum calculus.

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