Domination Number and Bondage Number of Lexicographic Product of Two Graphs

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Abstract- The domination number of a graph is the minimum number of vertices in a set S such that every vertex of the graph is either in S or adjacent to a member of S. The bondage number b(G) of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with a domination number greater than the domination number of G. In this paper, we study the domination number and bondage number of the Lexicographic product of two paths, Lexicographic product of path and a graph with given maximum degree.

Keywords – Graph, Lexicographic product, Domination number, Bondage number.

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I. INTRODUCTION

Unless mentioned otherwise for terminology and notation the reader may refer F. Harary [3], new ones will be introduced as and when found necessary.

Let $G = (V(G), E(G))$ be a finite, simple and connected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The neighborhood of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. Denote $E_G(v)$ to be the set of edges incident with v in G. The closed neighborhood of a vertex v in a graph G is $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v denoted by $d_G(v)$ is the cardinality of $N_G(v)$. Denote $\delta(G)$ and $\Delta(G)$ to be the minimum and maximum degree of G, respectively. A vertex of degree zero is called an isolated vertex. An edge incident with a vertex of degree one is called a pendant edge. A subset $S \subseteq V(G)$ of vertices is a dominating set if every vertex in $V(G)-S$ is adjacent to at least one vertex of S. The domination number $\gamma(G)$ is the minimum cardinality of all dominating sets in G. The domination is such an important concept that it has become one of the most widely studied topics in graph theory and also is frequently used to study property of networks. For a detailed survey of domination one can see [7], [8] and [9]. Graphs with domination numbers changed upon the removal of an edge were first investigated by Walikar and Acharya [12] in 1979. A graph is called edge-domination-critical graph if $\gamma(G - e) > \gamma(G)$ for every edge e in G. The edge-domination-critical graph was characterized by Bauer et al. [1] in 1983; that is, a graph is edge- domination-critical if and only if it is the union of stars. However, for lots of graphs, the domination number is out of the range of one-edge removal. It is immediate that $\gamma(H) \geq \gamma(G)$ for any spanning subgraph H of G. Every graph G has a spanning forest T with $\gamma(G) = \gamma(T)$, and so, in general, a graph has a nonempty subset $F \subseteq E(G)$ for which $\gamma(G - F) = \gamma(G)$.

A measure of the efficiency of a domination in graphs was first given by Bauer et al. [1] in 1983, who called this measure as domination linestability, defined as the minimum number of lines (i.e., edges) which when removed from G increases γ.

In 1990, Fink et al. [2] formally introduced the bondage number as a parameter for measuring the vulnerability of the interconnection network under link failure. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with the minimum cost. So we must consider whether its function remains good when the network is attacked. Suppose that someone such as a saboteur does not know which sites in the network take part in the domination role, but does know that the set of these special sites corresponds to a minimum dominating set in the related graph. Then how many links does he has to attack so that the cost cannot remains the same in order to dominate the whole network? That minimum number of links is just the bondage number. The bondage number b(G) of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$, that is,

$b(G) = \min \{ |B| \setminus B \subseteq E(G), \gamma(G - B) > \gamma(G) \}$.

Fink et al. [2] computed the exact value of the bondage number of cycles, paths and complete multipartite graphs and showed that $b(T) \leq 2$ for any tree T. Hartnell and Rall [4] characterize trees with bondage number 2.

Hartnell and Rall [5] proved that for the cartesian product $G_n = K_n \square K_n$, $n > 1$, we have $b(G_n) = \frac{3}{4}\Delta(G)$

Definition 1.1. Given graphs G and H, the lexicographic product G[H] has vertex set {(g, h) : g ∈ V (G), h ∈ V (H)} and two vertices (g, h), (g', h') are adjacent if and only if either [g, g'] is an edge of G or g = g' and [h, h'] is an edge of H.

II. DOMINATION NUMBER OF LEXICOGRAPHIC PRODUCT OF TWO GRAPHS

In the following results we give the domination number of Lexicographic product of path and graph.

Theorem 2.1. If G is a graph of order $m \geq 2$ with $\Delta(G) = m - 1$ then $\gamma(P_n[G]) = \left\lceil \frac{n}{3} \right\rceil$, $n \geq 2$.

Proof . Let G be a graph of order m and $v_k \in V(G)$ be a vertex of degree $\Delta(G) = m - 1$. Let $P_n : u_1, u_2, \ldots, u_n$ be a path on n vertices and $G_1, G_2, G_3, \ldots, G_n$ be the n copies of the graph G , substituted in the places of $u_1, u_2, u_3, \ldots, u_n$, respectively, in the lexicographic product $P_n[G]$, as shown in Figure 1.

In $P_n[G]$, let $x_i = (u_1, v_k) \in G_i$, $1 \leq i \leq n$ be the copies of $v_k \in G$. For $2 \leq i \leq n - 1$, every vertex of G_i is adjacent to every vertex of G_{i-1} and G_{i+1} only. Hence, $\gamma(P_n[G]) \geq \gamma(P_n)$. We prove the result in the following four cases.

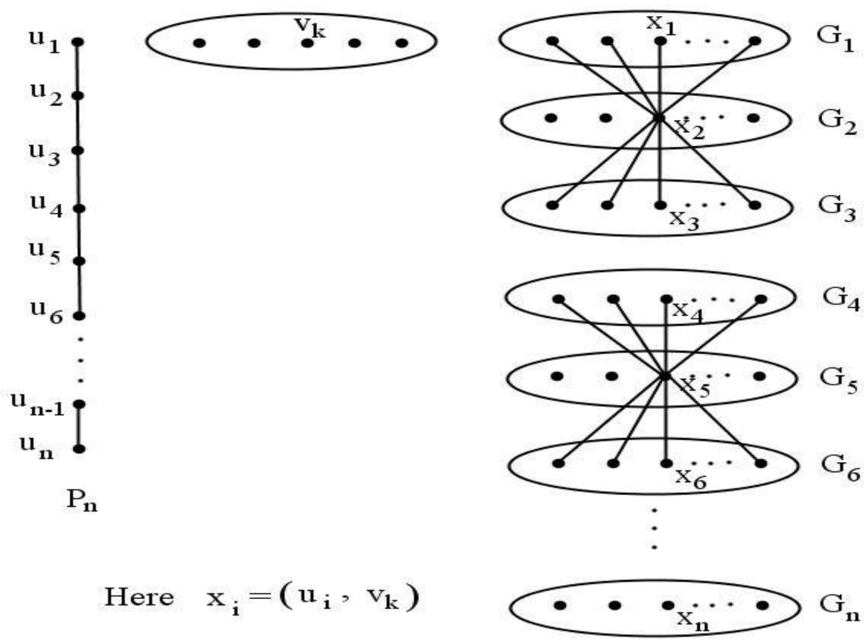

Figure 1 Domination number of $P_n[G]$, $\Delta(G) = m - 1$

Case(i): $n = 2$.

Let $P_2 : u_1, u_2$ be a path on two vertices and G be any graph on $m \geq 2$ vertices, labeled as $v_1, v_2, \ldots, v_m$.

From Figure 2, clearly, the vertex $x_1 \in G_1$ dominates all the vertices of $P_2[G]$. Hence, $\gamma(P_2[G]) = 1$.

Case(ii): $n = 3k$, $k \geq 1$.

In this case, the set $D = \{x_{3t-1} / 1 \leq t \leq k\}$ is the minimum dominating set of $P_n[G]$. Hence,

$\gamma(P_n[G]) = k = \left\lceil \frac{n}{3} \right\rceil$, where $n = 3k$, $k \geq 1$.

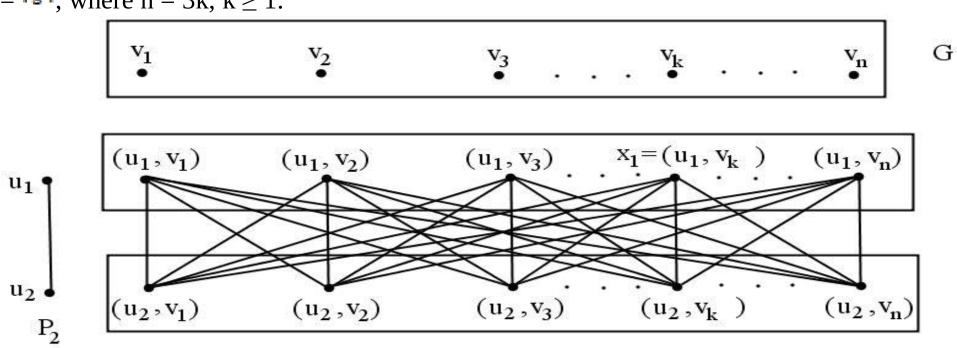

Figure 2. Domination number of P2[G]

Case(iii): $n = 3k + 1, k \geq 1$.
In this case, the set $D = \{x_{3t-1} / 1 \leq t \leq k\} \cup \{x_{3k}\}$ is the minimum dominating set of $P_n[G]$. Hence,

$\gamma(P_n[G]) = k + 1 = \left\lceil \frac{n}{3} \right\rceil$, where $n = 3k + 1, k \geq 1$.

Case(iv): $n = 3k + 2, k \geq 1$.
In this case, the set $D = \{x_{3t-1}/1 \leq t \leq k\} \cup \{x_{3k+1}\}$ is the minimum dominating set of $P_n[G]$. Hence,

$\gamma(P_n[G]) = k + 1 = \left\lceil \frac{n}{3} \right\rceil$, where $n = 3k + 2, k \geq 1$.

Theorem 2.2. For any graph G of order $m \geq 4$ with $\Delta(G) < m - 1$, $\gamma(P_2[G]) = 2$

Proof . Let $P_2 : u_1, u_2$ be a path on two vertices and G be any graph on $m \geq 4$ vertices, labeled as $v_1, v_2, \ldots, v_m$.

The lexicographic product $P_2[G]$, where G is a graph of order $m \geq 4$, as shown in Figure 3, is a graph on 2m vertices. The vertices (u_1, v_1), (u_1, v_m), (u_2, v_1) and (u_2, v_m) are of degree n + 1 and all other vertices are of degree n + 2. Therefore, $\gamma(P_2[G]) > 1$. The vertices (u_1, v_3) and (u_2, v_3) dominates all the vertices of $P_2[G]$. Hence, $\gamma(P_2[G]) = 2$, $m \geq 4$.

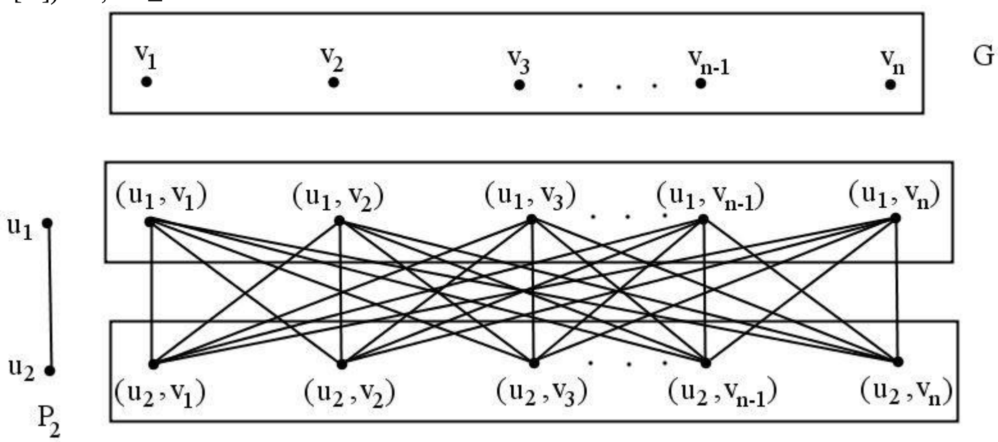

Figure 3. Domination number of $P_2[G]$ with $|V(G)| \geq 4$

Theorem 2.3. For any graph G of order $m \geq 4$ with $\Delta(G) < m - 1$, $\gamma(P_3[G]) = 2$

Proof . Let G be a graph on $m \geq 4$ vertices with $\Delta(G) < m - 1$.

Let $P_3 : u_1, u_2, u_3$ be a path on three vertices and G be any graph on $m \geq 4$ vertices with $\Delta(G) < m - 1$. Let the vertices of G be labeled as $v_1, v_2, \ldots, v_m$.

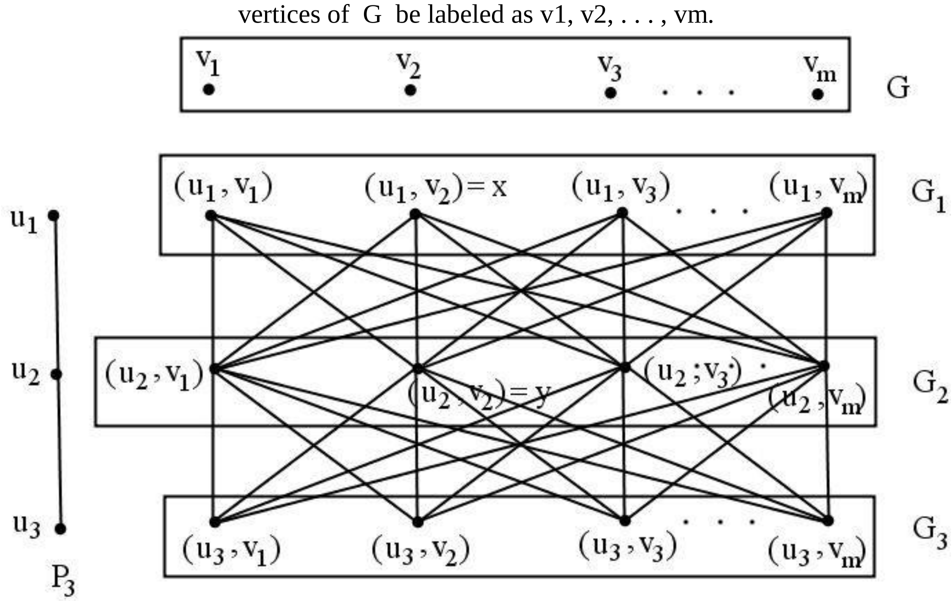

Figure 4. Domination number of $P_3[G]$

Let G1, G2, G3 be the copies of G substituted in the places of u1, u2, u3, respectively, in the lexicographic product P3[G], where |V (G)| ≥ 4, as shown in Figure 4. Since, the degree of every vertex in P3[G] is less than |V (P3[G])| −1, no single vertex can dominate all the vertices, i.e., γ(P3[G]) > 1. Any vertex, say, x in G1 dominates all the vertices in G2 and any vertex, say, y in G2 dominates all the vertices of G1 and G3. Therefore, the set {x, y} is the minimum dominating set. Hence, γ(P3[G]) = 2, m ≥ 4.

Theorem 2.4. If G is a graph of order m ≥ 4 with Δ(G) < m – 1 then

$$\gamma(P_n[G]) = \begin{cases} 2k, & if\ n = 4k, k \geq 1 \\ 2k+1, & if\ n = 4k+1, k \geq 1 \\ 2k+2, & if\ n = 4k+2, k \geq 1 \\ 2k+2, & if\ n = 4k+3, k \geq 1 \end{cases}$$

Proof. Let G be a graph of order m with Δ(G) < m −1. Let G1, G2, G3, . . ., Gn be the copies of the graph G, substituted in the places of u1, u2, u3, . . ., un, respectively, in the lexicographic product Pn[G], as shown in Figure 5.

In Pn[G], let x1 ∈ G1, x2 ∈ G2, . . ., xn ∈ Gn be the copies of vk ∈ G. Here four cases arise.

Case(i): n = 4k, k ≥ 1.

In this case, the set of vertices, D = {x4t−2/1 ≤ t ≤ k}∪{x4t−1/1 ≤ t ≤ k} form a minimum dominating set with cardinality 2k. Thus γ(Pn[G]) = 2k, where n = 4k, k ≥ 1.

Case(ii): n = 4k + 1, k ≥ 1.

In this case, the set of vertices, D = {x4t−2/1 ≤ t ≤ k} ∪ {x4t−1/1 ≤ t ≤ k} ∪ v4k form a minimum dominating set with cardinality 2k + 1. Thus, γ(Pn[G]) = 2k + 1, where n = 4k + 1, k ≥ 1.

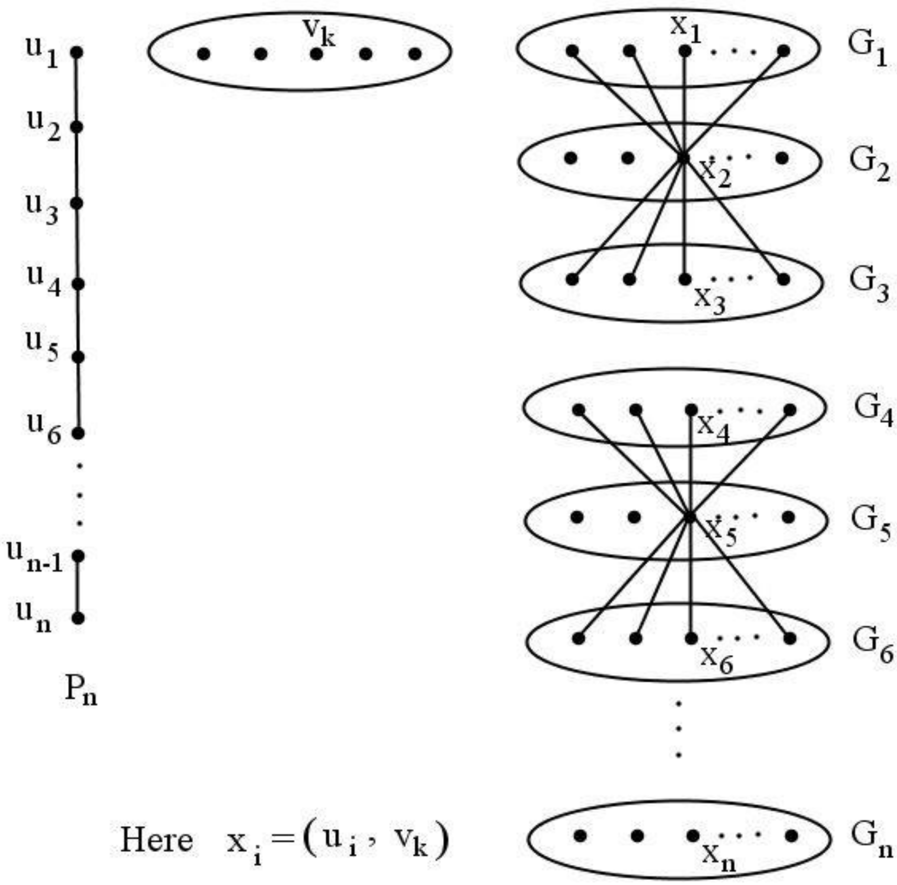

Figure 5. Domination number of Pn[G], Δ(G) < m −1

Case(iii): n = 4k + 2, k ≥ 1.

In this case, the set of vertices, D = {x4t−2/1 ≤ t ≤ k} ∪ {x4t−1/1 ≤ t ≤ k} ∪ {x4k+1, x4k+2} form a minimum dominating set with cardinality 2k + 2. Thus, γ(Pn[G]) = 2k + 2, where n = 4k + 2, k ≥ 1.

Case(iv): $n = 4k + 3, k \geq 1$.
In this case, the set of vertices, $D = \{x_{4t-2}/1 \leq t \leq k\} \cup \{x_{4t-1}/1 \leq t \leq k\} \cup \{x_{4k+2}, x_{4k+3}\}$ form a minimum dominating set with cardinality $2k + 2$. Thus $\gamma(P_n[G]) = 2k + 2$, where $n = 4k + 3, k \geq 1$.

III. BONDAGE NUMBER OF LEXICOGRAPHIC PRODUCT OF TWO GRAPHS

Theorem 3.1. If a graph G of order m has at most one vertex of degree $m-1$ then $b(P_2[G]) = 1$.
Proof . Let a graph G of order m has at most one vertex, say, v_k of degree $m-1$. The lexicographic product $P_2[G]$ is as shown in Figure 6.

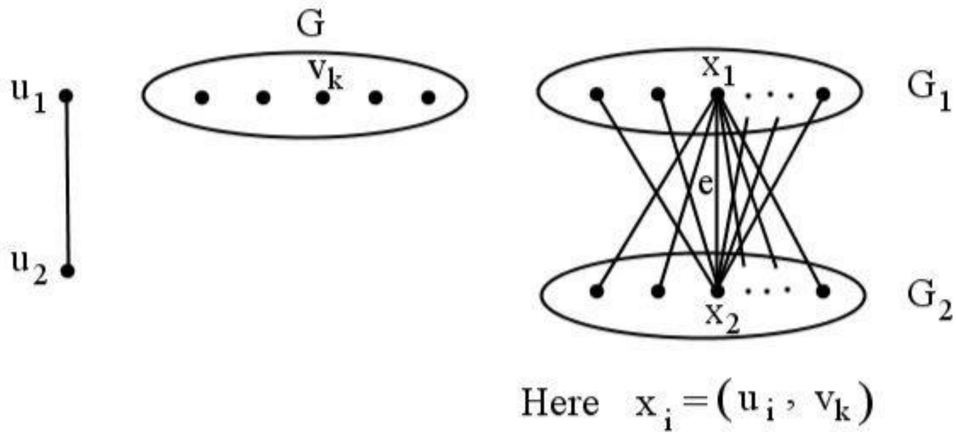

Figure 6. Bondage number of P2[G]

Let G_1 and G_2 be the copies of G substituted in the places of u_1 and u_2, respectively, in $P_2[G]$. From Figure 6, clearly, the vertex $(u_1, v_k) \in G_1$ dominates all the vertices of G_1 and G_2. Also, the vertex $(u_2, v_k) \in G_2$ dominates all the vertices of G_1 and G_2. Hence, $\gamma(P_2[G]) = 1$. The singleton sets $\{x_1\}$ and $\{x_2\}$ are the only two minimum dominating sets in $P_2[G]$. Removal of the edge e between the vertices x_1 and x_2 makes the vertex x_1 undominated by x_2 and the vertex x_2 undominated by x_1. Therefore, $\gamma(P_2[G] - e) > \gamma(P_2[G])$. Hence, $b(P_2[G]) = 1$.
Theorem 3.2. If a graph G of order m has at most one vertex of degree $m - 1$ then $b(P_3[G]) = 1$.
Proof. Let a graph G of order m has at most one vertex, say, v_1 of degree $m-1$.

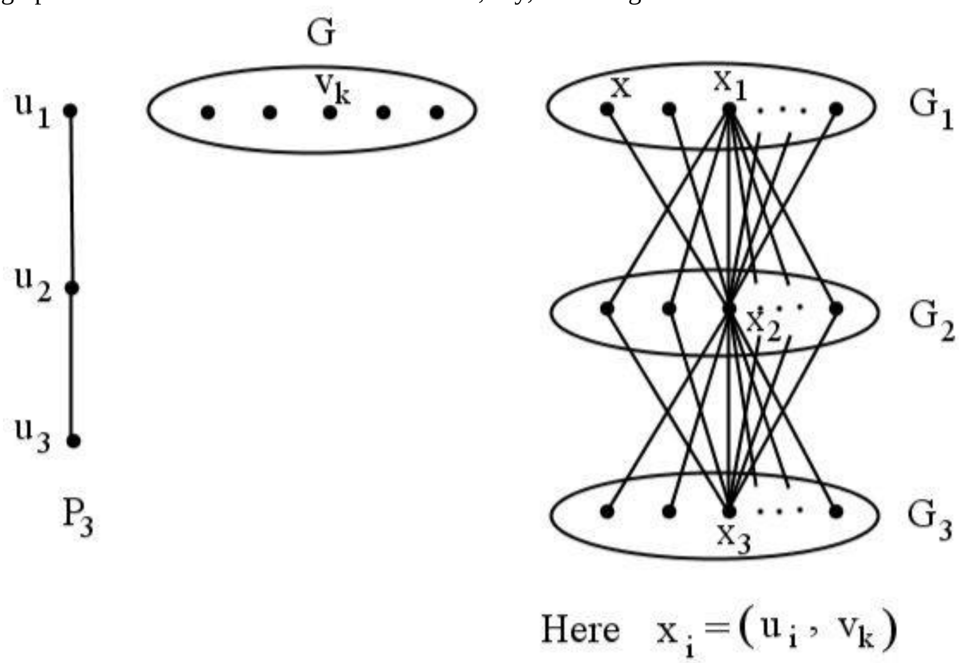

Figure 7. Bondage number of P3[G]

Let G1, G2 and G3 be the copies of G substituted in the places of u1, u2 and u3, respectively, in the lexicographic product P3[G], as shown in Figure 7. Let x1 ∈ G1, x2 ∈ G2 and x3 ∈ G3 be the copies of vk, in the lexicographic product P3[G]. Clearly, x2 is the only vertex which dominates all the vertices of G1, G2 and G3. Hence, γ(P3[G]) = 1. Removal of any edge incident with x2 from P3[G], increases the domination number, i.e., γ (P3[G] − x2 x) > γ (P3[G]). Hence, b(P3[G]) = 1.

Theorem 3.3. If G is a graph of order m and having at most one vertex of degree m −1 then b(P4[G]) = m + 1.

Proof. Let G be a graph of order m and having at most one vertex, say, v1 of degree m −1. The lexicographic product P4[G] is as shown in Figure 8. Let G1, G2, G3 and G4 be the copies of G substituted in the places of u1, u2, u3 and u4, respectively, in the lexicographic product P4[G].

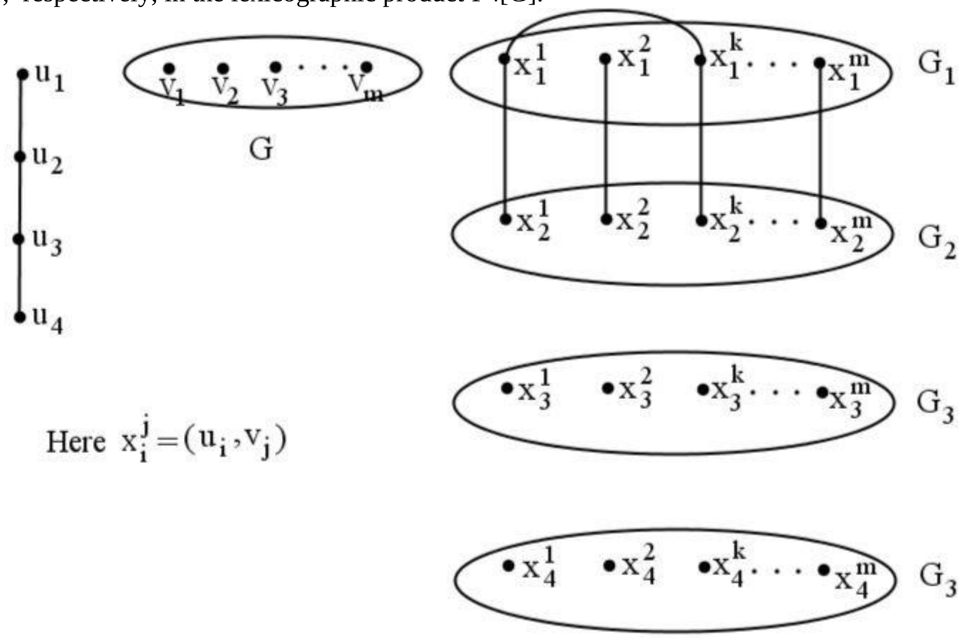

Figure 8. Bondage number of P4[G]

Let $V(G_1) = \{x_1^i / 1 \le i \le m\}$,
$V(G_2) = \{x_2^i / 1 \le i \le m\}$,
$V(G_3) = \{x_3^i / 1 \le i \le m\}$,
$V(G_4) = \{x_4^i / 1 \le i \le m\}$
and E(Gi), represent a set edges in Gi and E(Gi − Gi+1) represent a set of edges between Gi and Gi+1.

For i = 2, 3, every vertex of Gi is adjacent to every vertex of Gi−1 and Gi+1 only. Hence, γ(P4(G)) = 2.

We first prove that, the removal of m edges from P4[G] does not increase the domination number.

Let F be a set of any m edges in P4[G].

Case(i): F ⊆ E(Gi), 1 ≤ i ≤ 4.

Subcase(i): F ⊆ E(G1).

Here, x_2^1 and x_4^1 dominates all the vertices of P4[G] −F .

Subcase(ii): F ⊆ E(G2).

Here, $x_1^1$and x_3^1 dominates all the vertices of P4[G] −F .

Subcase(iii): F ⊆ E(G3).

Here, x_2^1 and x_4^1 dominates all the vertices of P4[G] −F .

Subcase(iv): F ⊆ E(G4).

Here, x_3^1 and x_1^1 dominates all the vertices of P4[G] −F .

Case(ii): F ⊂ E(Gi − Gi+1), 1 ≤ i ≤ 3.

Subcase(i): F ⊂ E(G1 − G2).

Here, x_1^1 and x_3^1 dominates all the vertices of P4[G] −F .

Subcase(ii): F ⊂ E(G2 − G3).

Here, x_2^1 and x_4^1 dominates all the vertices of P4[G] −F .

Subcase(iii): F ⊂ E(G3 − G4).
Here, x_4^1 and x_2^1 dominates all the vertices of P4[G] −F .
Case(iii): Suppose F contains edges from at least two copies of G in the lexicographic product.
There exist a vertex x_2^1 in G2, which dominates all the vertices of G1 and G2. Also, there exists a vertex x_3^1 in G3, which dominates all the vertices of G3 and G4. Hence, $\{x_2^1, x_3^1\}$ is the minimum dominating set, i.e., γ(P4[G]) = 2.
Case(iv): Suppose F contains edges from at least two copies of G and edges from E(Gi − Gi+1), i = 1, 2, 3 in the lexicographic product.
There exist a vertex x_2^i in G2, which dominates all the vertices of G1 and G3. Also, there exists a vertex x_3^j in G3, which dominates all the vertices of G2 and G4. Hence, $\{x_2^i, x_3^j\}$ is the minimum dominating set, i.e.,γ(P4[G]) = 2.
The set of edges T = $\{x_1^i x_2^i / 1 \le i \le m\} \cup x_1^1 x_1^k$ where x_1^1 is the vertex in V (G1) with degree m −1 and x_1^k is any vertex in V (G1), is the smallest set such that γ(P4[G] − T) > γ(P4[G]). Hence, b(P4[G]) = m + 1.

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