New Types of semi-open sets

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Abstract- Velicko investigated the concepts of θ-open and δ-open sets. Stone studied the concepts of regular open sets and Levine introduced the concept of semi-open sets. In this paper several versions of semi-open sets have been introduced by mixing the concepts of θ-Open sets, δ-open sets and regular open sets and their basic properties have been discussed.
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The notions of θ-open sets and δ-open sets were studied by Velicko[6]. Levine[3] discussed the concept of semi-open sets to analyze the functions weaker than continuous functions. In this paper, new types of semi-open sets have been introduced by using the concepts of θ-open sets and δ-open sets.

I. PRELIMINARIES

Throughout this paper (X, τ) is a topological space and S is a subset of X. The interior and closure of a subset S of X with respect to the topology τ are denoted by IntS and ClS respectively. S is regular open [5] if S=Int Cl S and is regular closed if S=Cl IntS. Let RO(X, τ) denote the collection of all regular open sets in (X, τ). Since the union of regular open sets is not regular open, RO(X,τ) is not topology on X. However, since the intersection of two regular open sets is regular open, RO(X, τ) is a base for some topology on X. This topology is denoted by τδ and the members of τδ are called δ-open sets in (X, τ). Clearly S is δ-open iff for all $x \in S$ there exists a closed set F with $x \in IntF \subseteq S$. That is every δ-open set is a union of regular open sets. The topology τδ is called the semi-regularization of τ. Clearly $RO(X,\tau) \subseteq \tau\delta \subseteq \tau$. Velicko defined the θ-closure operator in a topological space (X, τ). An element x in X is in θ-closure of a subset A of X, denoted by $x \in \theta ClA$ if for every open set U containing x, $ClU \cap A \neq \emptyset$. The θ-interior operator can be defined as $\theta IntA = X \backslash \theta Cl(X \backslash A)$. The subset A of X is θ-closed if A = θClA and is θ-open if A = θIntA. Clearly A is θ-open if and only if X\A is θ-closed. In fact S is θ-open if for all $x \in S$ there exists an open set U with $x \in U \subseteq ClU \subseteq S$. The collection of all θ-open sets in (X, τ) is a topology on X, denoted τθ. It is noteworthy to see that $\tau\theta \subseteq \tau\delta \subseteq \tau$. Mila Mrsevic and Andrijevic [2] concluded that the operator θCl is not idempotent that is $\theta Cl(\theta ClA) \neq \theta ClA$. A subset S of X is semi-open [3] if $S \subseteq ClIntS$ and is semi-closed if $S \supseteq ClIntS$. In the next two sections, the new versions of sem-iopen sets have been introduced by using the operators namely IntδA, ClδA, IntθA and ClθA which respectively denote the δ-interior, δ-closure , the θ-interior and θ-closure of A in (X, τ). The following definitions and results will be useful in sequel.

Definition 1.1: A space (X, τ) is
(i).regular if for every x in X and for every open set V containing x there exists an open set U such that $x \in U \subseteq ClU \subseteq V$.
(ii).almost regular [4] if for every x in X and for every open set V containing x there exists an open set U such that $x \in U \subseteq ClU \subseteq IntClV$.
The next lemma has been established in [4].

Lemma 1.2: A topological space (X,τ) is (i) regular if and only if τθ = τ; (ii) semi-regular if and only if τδ = τ and (iii) almost regular if and only if τθ = τδ .

The following definition is due to the authors [1].

Definition 1.3:A subset A of a space (X, τ) is
(i). regular θ-open (resp. regular θ-closed) in (X,τ) if A= Intθ ClθA (resp. A= Clθ IntθA).
(ii). regular δ-open (resp. regular δ-closed) in (X,τ) if A= Intδ ClδA (resp. A= Clδ IntδA).
(i). regular θ*-open (resp. regular θ*-closed) in (X,τ) if A= Intθ ClA (resp. A= Clθ IntA).

(ii). regular δ*-open (resp. regular δ*-closed) in (X,τ) if A= Intδ ClA (resp. A= Clδ IntA).

II. SEMI θ-OPEN SETS AND SEMI θ*-OPEN SETS

In this section, the notions of semi θ-open, semi θ-closed, semi θ*-open and semi θ-closed sets are introduced and studied. Before this , θ-open and θ-closed are characterized in the following proposition by using semi-open sets.

Proposition 2.1: A subset A of a space (X, τ) is

(i).θ-open iff for all $x\in A$ there exists a semi-open set U with $x\in$ Int $U\subseteq$ Cl $U\subseteq A$.

(ii).θ-closed iff for all $x\notin A$ there exists a semi-closed set F with $A\subseteq$ Int $F\subseteq$ Cl F and $x\notin$ Cl F.

Proof: The necessary part of (i) follows easily from the definition of a θ-open as every open set is semi-open . Let A be subset of X. Suppose for all $x\in A$ there exists a semi-open set U with $x\in IntU\subseteq ClU\subseteq A$. Since IntU is an open set and since $x\in IntU\subseteq Cl\ IntU\subseteq ClU\subseteq A$ it follows that A is θ-open. This proves (i). Now suppose A is θ-closed. Then X\A is θ-open. Then for all $x\in X\backslash A$ there exists a semi-open set U with $x\in IntU\subseteq U\subseteq ClU\subseteq X\backslash A$ that is $A\subseteq X\backslash ClU\subseteq X\backslash U\subseteq X\backslash IntU = Cl(X\backslash U)$ that implies $A\subseteq IntF\subseteq ClF$ and $x\notin Cl\ F$ where F = X\U which is semiclosed. Conversely let for all $x\notin A$ there exists a semi-closed set F with $A\subseteq IntF\subseteq Cl\ F$ and $x\notin$ Cl F. If $x\in X\backslash A$ then there exists a semi-closed set F with $A\subseteq Int\ F\subseteq Cl\ F$ and $x\notin Cl\ F$ that implies $x\in X\backslash Cl\ F\subseteq X\backslash F\subseteq X\backslash IntF\subseteq X\backslash A$ that is $x\in Int(X\backslash F)\subseteq X\backslash F\subseteq Cl(X\backslash F)\subseteq X\backslash A$ which further implies by(i) that X\A is θ-open. Therefore A is θ-closed. This proves (ii).

Definition 2.2: A subset A of X

(i). is semi θ-open (resp. semi θ*-open) in (X,τ) if $A\subseteq Cl\theta Int\theta\ A$ (resp. $A\subseteq Cl\theta$ Int A).

(ii). is semi θ-closed (resp.semi θ*-closed) in (X,τ) if $A\supseteq Int\theta\ Cl\theta\ A$ (resp. $A\supseteq Int\theta$ Cl A)

Proposition 2.3: A is semi θ-open in (X,τ) if and only if A is semi-open in $(X,\tau\theta)$

Proof: Follows from the fact that $(X,\tau\theta)$ is a topological space.

Remark 2.4: Since $\tau\theta$ is a topology on X, all the properties of semi-open sets in (X,τ) also hold for semi θ-open sets in (X,τ) as seen in the next three propositions.

Proposition 2.5:

(i).If A is regular θ-open (resp. θ-open) then it is semi θ-closed(resp. semi θ-open).

(ii).If A is regular θ-closed (resp. θ-closed) then it is semi θ-open(resp. semi θ-closed)

Proposition 2.6:

(i).If A is semi-open or semi θ-open or regular θ*-closed in (X,τ) then it is semi θ*-open .

(ii).If A is semi-closed or semi θ-closed or regular θ*-open in (X,τ) then it is semi θ*-closed.

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Proposition 2.7:

(i).If A is semi θ*-open then it is contained in a regular θ*-closed set.

(ii).If A is semi θ*- closed then it is contained in a regular θ*-set.

III. SEMI δ-OPEN SETS AND SEMI δ*-OPEN SETS

In this section , semi δ-open sets and semi δ*-open sets are introduced and studied. Before this , δ-open and δ-closed sets are characterized in the next proposition by using semi-open sets.

Proposition 3.1: A subset A of a space (X, τ) is

(i).δ-open iff for all $x\in A$ there exists a semi-closed set F with $x\in$ Int $F\subseteq A$

(ii).δ-closed iff for all $x\notin A$ there exists an open U with $A\subseteq$ Cl U and $x\notin$ Cl U.

Proof: The necessary parts of (i) follows easily from the definition of a δ-open as every closed set is semi-closed. Suppose for all $x\in A$ there exists a semi-closed set F with $x\in Int\ F\subseteq A$. If $x\in A$ then there exists a semi-closed set F with $x\in Int\ F\subseteq A$ that implies $x\in Int\ F = Int\ sCl\ F\subseteq Int\ ClF\subseteq A$ that further implies that $x\in IntB\subseteq A$ where B = ClF is closed. Therefore A is δ-open. This proves (i). Now suppose A is δ-closed. Then X\A is δ-open. Then for $x\in X\backslash A$ there exists a closed set F with $x\in Int\ F\subseteq X\backslash A$ that is $A\subseteq X\backslash IntF = Cl(X\backslash F)$ that implies $A\subseteq Cl\ U$ and $x\notin ClU$ where U = X\F which is open. Conversely let for all $x\notin A$ there exists an open set U with $A\subseteq ClU$ and $x\notin ClU$. If

$x \in X \backslash A$ then there exists an open set U with $A \subseteq ClU$ and $x \notin ClU$ that implies $x \in X \backslash ClU \subseteq X \backslash A$ that is $x \in Int(X \backslash U) \subseteq X \backslash A$ which further implies by (i) that $X \backslash A$ is δ-open. Therefore A is δ-closed. This proves (ii).

Definition 3.2: A subset A of a space (X, τ) is
(i).semi δ-open (resp semi δ*-open) in (X, τ) if $A \subseteq Cl\delta Int\delta A$ (resp. $A \subseteq Cl\delta IntA$) and
(ii). semi δ-closed (resp semi δ*-closed)in (X,τ) if $A \supseteq Int\delta Cl\delta A$ (resp. $A \supseteq Int\delta ClA$).

Proposition 3.3: A is semi δ-open (resp. semi δ-closed) in (X,τ) if and only if A is semi-open (resp. semi-closed)in (X,τδ).

Proof: Follows from the fact that (X,τδ) is a topological space.

Remark 3.4: Since τδ is a topology on X, all the properties of semi-open sets in (X,τ) also hold for semi δ-open sets in (X,τ) as seen in the next three propositions

Proposition 3.5:
(i).If A is regular δ-open (resp. δ-open) then it is semi δ-closed (resp. semi δ-open).
(ii).If A is regular δ-closed (resp. δ-closed) then it is semi δ-open(resp. semi δ-closed).

Proposition 3.6:
(i).If A is semi-open or semi δ-open or regular δ*-closed in (X,τ) then it is semi δ*-open.
(ii).If A is semi-closed or semi δ-closed or regular δ*-open in (X,τ) then it is semi δ*-closed.
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Proposition 3.7:
(i).If A is semi δ-open (resp.semi δ*-open) then it is contained in a regular δ-closed (resp. regular δ*-closed)set.
(ii).If A is semi δ-closed (resp.semi δ*- closed) then it is contained in a regular δ- open (resp.regular δ*- open) set.

4. Some common properties
Proposition 4.1:
(i).A is semi θ-open (resp. semi θ*-open) iff X\A is semi θ-closed (resp. semi θ*-closed)
(ii).A is semi δ-open (resp. semi δ*-open) iff X\A is semi δ-closed (resp. semi δ*-closed)

Proposition 4.2:
(i) The intersection of two semi θ-open (resp. semi δ-open, resp. semi θ*-open, resp. semi δ*-open)sets in (X,τ) is not semi θ-open (resp. semi δ-open, resp. semi θ*-open, resp. semi δ*-open) in (X,τ) .
(ii) The union of two semi θ-closed (resp. semi δ-closed, resp. semi θ*-closed, resp. semi δ*-closed)sets in (X,τ) is not semi θ-closed (resp. semi δ-closed, resp. semi θ*-closed, resp. semi δ*-closed) in (X,τ).
(iii) The intersection of two semi θ-closed (resp. semi δ-closed, resp. semi θ*-closed, resp. semi δ*-closed)sets in (X,τ) is semi θ- closed (resp. semi δ-closed, resp. semi θ*-closed, resp. semi δ*-closed) in (X,τ).
(iv) The union of two semi θ-open (resp. semi δ-open, resp. semi θ*-open, resp. semi δ*-open)sets in (X,τ) is semi θ- open (resp. semi δ-open, resp. semi θ*-open, resp. semi δ*-open) in (X,τ)
The next three theorems follow from Lemma 1.2.

Theorem 4.3: In an almost regular space
(i).A is semi θ-open iff it is semi δ-open,
(ii). A is semi θ*-open iff it is semi δ*-open,
(iii).A is semi θ-closed iff it is semi δ-closed,
(iv). A is semi θ*-closed iff it is semi δ*-closed,

Theorem 4.4: In a semi-regular space
(i).A is semi-open iff it is semi δ-open,
(ii).A is semi-closed iff it is semi δ-closed,

Theorem 4.5: In a regular space the following are equivalent.
(i).A is semi-open

(ii). A is semi δ-open
(iii). A is semi δ*-open
(ii). A is semi θ-open
(iii). A is semi θ*-open

IV. CONCLUSION:

The existing concepts that are related to θ-open and δ-open sets have been characterized in regular, semi-regular and almost regular spaces by the new notions such as semi θ-open , semi θ*-open, semi δ-open and semi δ*-open sets .

V. REFERENCES

[1] V. Amsaveni, M.Anitha and A.Subramanian, "Weak and strong forms of regular open sets", (submitted).
[2] Mila Mrsevic, D.Andrijevic , "On θ-connectedness and θ-closure spaces", Topology and its Applications, vol.123, pp.157-166, 2002.
[3] Norman Levine, "Semi open sets and semi continuity in topological spaces", Amer. Math. Monthly, vol.70, pp.36-41, 1963.
[4] T.G.Raghavan, "θ-closed subsets of topological spaces", Indian J. pure. appl. Math. vol.15, no.3, pp. 233, 1984.
[5] M.H.Stone, "Application of the theory of Boolean rings to general topology", Trans. Amer.Math. Soc., vol. 41, pp. 374-481, 1937.
[6] N.V.Velicko , "H-closed topological spaces", Amer. Math. Soc. Trans. Vol.78, no.2, pp.103-118, 1968.