Some new classes of Ideal convergent difference multiple sequence spaces of Fuzzy real numbers

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Abstract- In 1981 H. Kizmaz defined the difference sequence spaces for crisp set and in 2007 Sahiner *et al.* was first introduced the idea of triple sequence spaces. In this article, we introduce the classes of fuzzy real-valued multiple sequences ${}_3(c^{I(F)})(\Delta_m , p)$, ${}_3(c_0{}^{I(F)})(\Delta_m , p)$, ${}_3(c^{I(F)})^R(\Delta_{m,} p)$ and ${}_3(c_0{}^{I(F)})^R(\Delta_{m,} p)$ where $p = \langle p_{nlk} \rangle$ is a triple sequence of bounded strictly positive numbers. We study different topological properties of these spaces like completeness, solid, monotone, symmetricity convergence free etc. We prove some inclusion results also.

Keywords – Ideal convergence (*I*-convergence), Fuzzy real-valued triple sequence, multiple sequences, solid, monotone, symmetric, convergence free, sequence algebra etc.

I. Introduction

In 1965 Zadeh [32] introduced the concepts of fuzzy sets and fuzzy set operations and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. In fact the fuzzy set theory has become an area of active area of research in science and engineering for the last 40 years. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It extends the scope and results of classical mathematical analysis by applying fuzzy logic to conventional mathematical objects, such as functions, sequences and series etc. The ideas of fuzzy set theory have been used widely not only in many engineering applications, such as, computer programming [11], population dynamics [3], quantum physics [20], control of chaos [10], bifurcation of non-linear dynamical system [13], approximation theory [2] etc., but also in various branches of mathematics, such as, theory of metric and topological spaces [8], theory of linear systems [22], studies of convergence of sequences of functions [4,14]. While studying fuzzy topological spaces, we face many situations where we need to deal with convergence of fuzzy numbers.

Using the idea of fuzzy real numbers, different types of fuzzy real-valued sequence spaces have been introduced and studied by several mathematicians. The initial works on double sequences of real or complex terms are found in Bromwich [5]. Hardy [12] introduced the notion of regular convergence for double sequences of real or complex terms. Agnew [1] studied the summability theory of multiple sequences and obtained certain theorems which have already been proved for double sequences by the author himself. Móricz [21] extended statistical convergence from single to multiple real sequences and obtained some results for real double sequences. Şahiner *et. al*. [24] developed statistical convergence for triple sequences of real numbers. A. J. Dutta *et al.*[7], P. Kumar *et al.*[18], E. Savas and A. Esi [26], are a few to be named those who have introduced different types of triple sequence spaces.

II. Definitions and background

Throughout the article *N* and *R* denote the sets of natural, and real numbers respectively and w, c, c_0, ℓ_∞ denote the spaces of all, convergent, null and bounded sequences respectively.

A fuzzy real number *X* is a fuzzy set on R, i.e. a mapping $X : R \to L$ (= [0, 1]) associating each real number t with its grade of membership X (*t*). Every real number *r* can be expressed as a fuzzy real number $\bar{r}$ as follows:

$$\bar{r}(t) = \begin{cases} 1 & if \quad t = r \\ 0 & otherwise \end{cases}$$

The α-level set of a fuzzy real number X, $0 < \alpha \leq 1$ denoted by $[X]^\alpha$ is defined as

$$[X]^\alpha = \{t \in R : X(t) \geq \alpha\}.$$

A fuzzy real number *X* is called convex if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r)),$ where $s < t < r.$ If there exists $t_0 \in R$ such that $X(t_0) = 1,$ then the fuzzy real number X is called normal. A fuzzy real number *X* is said to be upper semi-continuous if for each $\varepsilon > 0,\ X^{-1}[0, a+\varepsilon)),$ for all $a \in L$ is open in the usual topology of *R*. The set of all upper semi continuous, normal, convex fuzzy number is denoted by *R(L*).

Let *D* be the set of all closed bounded intervals $X = [X^L, X^R]$ on the real line *R*. Then $X \leq Y$ if and only if $X^L \leq Y^L$ and $X^R \leq Y^R$.

Also let $d(X,Y) = \max(|X^L - X^R|, |Y^L - Y^R|).$ Then (D, d) is a complete metric space.

Let $\bar{d} : R(L) \times R(L) \to R$ be defined by $\bar{d}(X,Y) = \sup\limits_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha)$, for $X, Y \in R(L).$

Then $\bar{d}$ defines a metric on $R(L)$ and $(R(L), \bar{d})$ is a complete metric space.

In order to generalize the notion of convergence of real sequences, Kostyrko, Šalát and Wilczyński [18] introduced the idea of Ideal convergence for single sequences in 2000-2001. Later on it was further developed by Šalát *et. al*. [26], Das *et. al*. [6], Tripathy and Tripathy [30], Tripathy and Sen [29], Kumar and Kumar [19], Sen and Roy [27] and many others.

Let *X* be a non empty set. A non-void class $I \subseteq 2^X$ (power set of *X*) is said to be an ideal if *I* is additive and hereditary, *i.e.* if *I* satisfies the following conditions:

(*i*) $A, B \in I \Rightarrow A \cup B \in I$ and (*ii*) $A \in I$ *and* $B \subseteq A \Rightarrow B \in I.$

A non-empty family of sets $F \subseteq 2^X$ is said to be a filter on *X* if

(*i*) $\varnothing \notin F$

(*ii*) $A, B \in F \Rightarrow A \cap B \in F$

(*iii*) $A \in F$ and $A \subseteq B \Rightarrow B \in F$.

For any ideal I, there is a filter $F(I)$ given by

$$F(I)=\{K \subseteq N : N \setminus K \in I\}.$$

An ideal $I \subseteq 2^X$ is said to be *non-trivial* if $I \neq \varnothing$ and $X \notin I$.

A subset E of N is said to have density $\delta(E)$ if $\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$ exists.

The details about the ideals of $2^{N\times N}$ are introduced and investigated by Tripathy and Tripathy [29]. Throughout the article, the ideals of $2^{N\times N\times N}$ will be denoted by I_3. .

Example 1. Let $I_3(\rho) \subset 2^{N\times N\times N}$ *i.e.* the class of all subsets of $N\times N\times N$ of zero natural density. Then $I_3(\rho)$ is an ideal of $2^{N\times N\times N}$.

Throughout ${}_3(w^F), {}_3(\ell_\infty^F), {}_3(c^F), {}_3(c_0{}^F), {}_3(c^F)^R, {}_3(c_0{}^F)^R$ denote the spaces of all, bounded, convergent in Pringsheim's sense, null in Pringsheim's sense, regularly convergent and regularly null fuzzy real valued triple sequences respectively.

A triple sequence can be defined as a function $x : N\times N\times N \to R(C)$.

A fuzzy real valued triple sequence $X=\langle X_{nlk}\rangle$ is a triple infinite array of fuzzy real numbers X_{nlk} for all $n, l, k \in N$ and is denoted by $\langle X_{nkl}\rangle$ where $X_{nlk} \in R(L)$.

A fuzzy real-valued triple sequence $X=\langle X_{nlk}\rangle$ is said to be I_3-*convergent* to the fuzzy number X_0, if for all $\varepsilon>0,$ the set $\{(n,l,k)\in N\times N\times N : \bar{d}(X_{nlk}, X_0)\geq \varepsilon\}\in I_3$. We write $I_3-\lim X_{nlk}=X_0$.

A fuzzy real-valued triple sequence $X=\langle X_{nlk}\rangle$ is said to be I_3-*bounded* if there exists a real number μ such that the set $\{(n,l,k)\in N\times N\times N : \bar{d}(X_{nlk}, \bar{0}) > \mu\}\in I_3$.

A fuzzy real-valued triple sequence space E^F is said to be solid if $\langle Y_{nlk}\rangle \in E^F$ whenever $|Y_{nlk}|\leq |X_{nlk}|$ for all $n,l,k\in N$ and $\langle X_{nlk}\rangle \in E^F$.

A fuzzy real-valued triple sequence space E^F is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

A fuzzy real-valued triple sequence E^F is said to be symmetric if $S(X) \subset E^F$, for all $X \in E^F$, where $S(X)$ denotes the set of all permutations of the elements of $X=\langle X_{nlk}\rangle$

A fuzzy real-valued triple sequence space E^F is said to be sequence algebra if $\langle X_{nlk}\otimes Y_{nlk}\rangle \in E^F$, whenever $\langle X_{nlk}\rangle, \langle Y_{nlk}\rangle \in E^F$.

A fuzzy real-valued triple sequence space E^F is said to be convergence free if $\langle Y_{nlk}\rangle \in E^F$ whenever $\langle X_{nlk}\rangle \in E^F$. and $X_{nlk} = \bar{0}$ implies $Y_{nlk} = \bar{0}$.

In 1981, H. Kizmaz [16] the difference sequence spaces for crisp set and studied the spaces $l_\infty(\Delta), c(\Delta), c_0(\Delta)$. Later on Seva [ ] introduced the notion of difference sequences for fuzzy real numbers a studied different properties of the spaces. In 2006, this concept was further generalised by Tripathy and Esi [31] as $Z(\Delta_m) = \{ (x_k) \in w : (\Delta_m x_k) \in Z\}$ where $m \geq 0$ be an integer and $Z = c, c_0, l_\infty$ and $\Delta_m\, x_k = x_k - x_{k+m}$ for all $k \in N$. Thereafter Tripathy and B. A. Baruah introduced the difference sequences of fuzzy real numbers L. we introduced the following for triple sequence:

$$\Delta_m X_{nlk} = X_{nlk} - X_{n,\,l+m\,,k} - X_{n\,,l\,,k+m} + X_{n,\,l+m,\,k+m} - x_{n+m,\,l,\,k} + X_{n+m,\,l+m,\,k} + X_{n+m,\,l,\,k+m} - X_{n+m,\,l+m,\,k+m}$$

Let $p = \langle p_{nkl}\rangle$ be a triple sequence of bounded strictly positive numbers. We introduce the following triple sequence spaces:

$$_3(c^{I(F)})(\Delta_m, p) = \{X = \langle X_{nlk}\rangle \in {}_3(w^F) : I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, X_0)]^{p_{nlk}} = 0, \text{ for some } X_0 \in R(L)\},$$

$$_3({c_0}^{I(F)})(\Delta_m, p) = \{X = \langle X_{nlk}\rangle \in {}_3(w^F) : I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, \bar{0})]^{p_{nlk}} = 0\},$$

$$_3(\ell_\infty^{I(F)})(\Delta_m, p) = \{X = \langle X_{nlk}\rangle \in {}_3(w^F) : \text{there exists a real number } \mu > 0 \text{ such that the set}$$

$$\{(n,l,k) \in N \times N \times N : [\bar{d}(\Delta_m X_{nlk}, \bar{0})]^{p_{nlk}} > \mu\} \in I_3\}.$$

$\langle X_{nlk}\rangle \in_3 (c^{I(F)})^R(\Delta_m, p)$ if and only if $\langle X_{nlk}\rangle \in {}_3(c^{I(F)})(\Delta_m, p)$ and the following limits exist:

$$I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, I_k)]^{p_{nlk}} = 0, \text{ for each } k \in N, \text{ for some } I_k \in R(L),$$

$$I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, J_l)]^{p_{nLk}} = 0, \text{ for each } l \in N, \text{ for some } J_l \in R(L)$$

$$\text{and } I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, K_n)]^{p_{nlk}} = 0, \text{ for each } n \in N, \text{ for some } K_n \in R(L).$$

$\langle X_{nlk}\rangle \in {}_3({c_0}^{I(F)})^R(\Delta_m, p)$ if and only if $\langle X_{nlk}\rangle \in {}_3({c_0}^{I(F)})(\Delta_m, p)$ and the following limits exist:

$$I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, \bar{0})]^{p_{nlk}} = 0, \text{ for each } k \in N$$

$$I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, \bar{0})]^{p_{nlk}} = 0, \text{ for each } l \in N$$

$$\text{and } I_3 - \lim [\bar{d}(\Delta_m X_{nlk}, \bar{0})]^{p_{nlk}} = 0, \text{ for each } n \in N.$$

Let

$$_3(\ell_\infty^{(F)})(\Delta_m, p) = \{X = \langle X_{nlk}\rangle \in {}_3(w^F) : \sup_{n,l,k} [\bar{d}(\Delta_m X_{nlk}, \bar{0})]^{p_{nlk}} < \infty\}.$$

Then $_3(\ell_\infty^{(F)})(\Delta_m. p)$ is a complete metric space with respect to the metric ρ defined by

$\rho(X,Y)=\sup_{n\,l,k}\left\{\bar{d}\,(\Delta_m X_{nlk}\,,\Delta_m Y_{nlk})\right\}^{\frac{p_{nlk}}{M}}$ where $M=\max\,(1,H),\; H=\sup_{n,l,k} p_{nlk}<\infty.$

Also, we define the following sequence spaces :

$$_3(c^{I(F)})^{BP}(\Delta_m,\,p)={}_3(c^{I(F)})(\Delta_m,\,p)\cap{}_3(\ell_\infty^{(F)})(\Delta_m,\,p),$$

$$_3(c_0^{I(F)})^{BP}(\Delta_m,\,p)={}_3(c_0^{I(F)})(\Delta_m,\,p)\cap{}_3(\ell_\infty^{(F)})(\Delta_m,\,p),$$

$$_3(c^{I(F)})^{BR}(\Delta_m,\,p)={}_3(c^{I(F)})^{R}(\Delta_m,\,p)\cap{}_3(\ell_\infty^{(F)})(\Delta_m,\,p),$$

$$_3(c_0^{I(F)})^{BR}(\Delta_m,\,p)={}_3(c_0^{I(F)})^{R}(\Delta_m,\,p)\cap{}_3(\ell_\infty^{(F)})(\Delta_m,\,p).$$

For particular values of *p* and ideal *I*, these sequence spaces reduce to many well known sequence spaces.

Let $\langle X_{nlk}\rangle$ and $\langle Y_{nlk}\rangle$ be two fuzzy real valued triple sequences. Then we say that $X_{nlk}=Y_{nlk}$ for almost all *n, l and k* relative to I_3 (in short *a.a. n, l* & *k r* I_3) if the set

$$\{(n,l,k)\in N\times N\times N : X_{nlk}\neq Y_{nlk}\}\in I_3.$$

Note. Let $p=\langle p_{nkl}\rangle$ be a triple sequence of bounded strictly positive numbers and $H=\sup p_{nlk}<\infty$. Then for sequences $\langle a_{nkl}\rangle$ and $\langle b_{nkl}\rangle$ of complex numbers, we have the following inequality:

$$\left|a_{nkl}+b_{nkl}\right|^{p_{nkl}}\leq D(\left|a_{nkl}\right|^{p_{nkl}}+\left|b_{nkl}\right|^{p_{nkl}}),\ \text{where}\ D=\max(1,2^{H-1}).$$

We procure the following existing result.

Lemma 1. *If a sequence space* E^F *is solid, then it is monotone.*

For the crisp set case, one may refer to Kamthan and Gupta [15], p.53.

III. MAIN Result

Theorem 1. *The following statements are equivalent*:

(*i*) $\langle X_{nlk}\rangle\in{}_3(c^{I(F)})(\Delta_m,\,p).$

(*ii*) *There exists a sequence* $\langle Y_{nlk}\rangle\in{}_3(c^{F})(\Delta_m,\,p)$ *such that* $\Delta_m X_{nlk}=\Delta_m Y_{nlk}$ *for a.a. n, l* & *k r* I_3.

(*iii*) *There exists a subset* $M=\{(n_i,l_j,k_m)\in N\times N\times N : i,j,m\in N\}$ *of* $N\times N\times N$ *such that* $M\in F(I_3)$ *and* $\langle X_{n_i l_j k_m}\rangle\in{}_3(c^{F})(\Delta_m,\,p).$

Proof. (*i*) $\Rightarrow$ (*ii*). Let $\langle X_{nlk}\rangle\in{}_3(c^{I(F)})(\Delta_m\,,\ p).$ Then there exists $X_0\in R(L)$ such that $I_3-\lim\left[\bar{d}(\Delta_m X_{nlk}\,,X_0)\right]^{p_{nlk}}=0.$

So for any $\varepsilon > 0,$ we have the set

$$\left\{(n,l,k) \in N \times N \times N : \lim\left[\bar{d}(\Delta_m X_{nlk}, X_0)\right]^{p_{nlk}} \geq \varepsilon \right\} \in I_3.$$

Let us consider the increasing sequences $(T_j), (M_j)$ and (N_j) of natural numbers such that if $p > T_j$ $q > M_j$, and $r > N_j$. Then the set

$$\left\{ (n,l,k) \in N \times N \times N : n \leq p\,; l \leq q; k \leq r \text{ and} \left[\bar{d}(\Delta_m X_{nlk}, X_0)\right]^{p_{nlk}} \geq \frac{1}{j} \right\} \in I_3.$$

We define the sequence $\langle Y_{nlk} \rangle$ as follows:

$Y_{nlk} = X_{nlk}$ if $n \leq T_1$ or $l \leq M_1$ or $k \leq N_1$

Also for all (n, l, k) with $T_j < n \leq T_{j+1}$ or $M_j < l \leq M_{j+1}$ or $N_j < k \leq N_{j+1}$,

let $Y_{nlk} = X_{nlk}$ if $\left[\bar{d}(\Delta_m X_{nlk}, X_0)\right]^{p_{nlk}} < \frac{1}{j},$ otherwise $Y_{nlk} = X_0$.

We show that $\langle Y_{nlk} \rangle \in {}_3(c^F)(\Delta_m, p).$

Let $\varepsilon > 0.$ We choose j such that $\varepsilon > \frac{1}{j}.$

We see that for $n > T_j$ $l > M_j$ and $k > N_j$, $\left[\bar{d}(\Delta_m Y_{nlk}, X_0)\right]^{p_{nlk}} < \varepsilon.$

Hence $\langle Y_{nlk} \rangle \in {}_3(c^F)(\Delta_m, p).$

Next let $T_j < n \leq T_{j+1}, M_j < l \leq M_{j+1}$ and $N_j < k \leq N_{j+1},$ then the set

$$A = \left\{ (n, l, k) \in N \times N \times N : \Delta_m X_{nlk} \neq \Delta_m Y_{nlk} \right\}$$
$$\subseteq \left\{(n,l,k) \in N \times N \times N : \left[\bar{d}(\Delta_m X_{nlk}, X_0)\right]^{p_{nlk}} \geq \frac{1}{j}\right\} \in I_3.$$

This implies $A \in I_3$ and so $\Delta_m X_{nlk} = \Delta_m Y_{nlk}$ for *a.a. n, l* & *k r.* I_3.

(*ii*) ⇒ (*iii*). Suppose there exists a sequence $\langle Y_{nlk} \rangle \in {}_3(c^F)(\Delta_m, p)$ such that $X_{nlk} = Y_{nlk}$ for *a.a. n, l* & *k r.* I_3. Let $M = \left\{ (n, l, k) \in N \times N \times N : \Delta_m X_{nlk} = \Delta_m Y_{nlk} \right\}$. Then $M \in F(I_3).$

We can enumerate M as $M = \{(n_i, l_j, k_m) \in N \times N \times N : i, j, m \in N\},$ on neglecting the rows and columns those contain finite number of elements. Then $\left\langle X_{n_i l_j k_m} \right\rangle \in {}_3(c^F)(\Delta_m, p).$

(iii)* ⇒ *(i) From (iii), (i) follows immediately.

Theorem 2. *If* $H = \sup_{n,l,k} p_{nlk} < \infty,$ *then the classes of sequences* ${}_3(c^{I(F)})^{BP}(\Delta_m, p),$ ${}_3(c_0^{I(F)})^{BP}(\Delta_m, p)$ ${}_3(c^{I(F)})^{BR}(\Delta_m, p)$ *and* ${}_3(c_0^{I(F)})^{BR}(\Delta_m, p)$ *are complete metric spaces with respect to the metric* ρ *defined by*

$$\rho(X,Y) = \sup_{n,l,k}[\bar{d}(\Delta_m X_{nlk}, \Delta_m Y_{nlk})]^{\frac{p_{nlk}}{M}}, \quad \textit{where } M = \max(1, H).$$

Proof. Let us consider the space ${}_3(c^{I(F)})^{BP}(\Delta_m, p).$

Let $\langle X^{(i)} \rangle$ be a Cauchy sequence in ${}_3(c^{I(F)})^{BP}(\Delta_m, p) \subset {}_3(\ell_\infty^{(F)})(\Delta_m, p),$ where $X^{(i)} = \langle X_{nlk}{}^{(i)}. \rangle.$

Since ${}_3(\ell_\infty^{(F)})(\Delta_m, p)$ is complete, so there exists $X \in {}_3(\ell_\infty^{(F)})(\Delta_m, p)$ such that

$$\lim_{i\to\infty} X^{(i)} = X, \quad \text{where } X = \langle X_{nlk} \rangle.$$

We claim that $X \in {}_3(c^{I(F)})^{BP}(p).$

Since $\langle X^{(i)} \rangle$ is Cauchy, so for a given $0 < \varepsilon < 1,$ there exists $n_0 \in N$ such that

$$\rho\left(X^{(i)}, X^{(j)}\right) < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq n_0.$$

$$\Rightarrow \left[\bar{d}(\Delta_m X_{nlk}{}^{(i)}, \Delta_m X_{nlk}{}^{(j)})\right]^{\frac{p_{nlk}}{M}} < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq n_0.$$

$$\Rightarrow \bar{d}\,(\Delta_m X_{nlk}{}^{(i)}, \Delta_m X_{nlk}{}^{(j)}) < \left(\frac{\varepsilon}{3}\right)^{\frac{M}{p_{nlk}}}, \quad \text{for all } i, j \geq n_0.$$

Again since $X^{(i)}, X^{(j)} \in {}_3(c^{I(F)})^{BP}(\Delta_m, p),$ so there exists fuzzy numbers Y_i and Y_j such that the sets

$$A = \left\{(n,l,k) \in N \times N \times N : \left[\bar{d}(\Delta_m X_{nlk}{}^{(i)}, Y_i)\right]^{p_{nlk}} < \left(\frac{\varepsilon}{3}\right)^M \right\} \in F(I_3)$$

And

$$B = \left\{(n,l,k) \in N \times N \times N : \left[\bar{d}(\Delta_m X_{nlk}{}^{(j)}, Y_j)\right]^{p_{nlk}} < \left(\frac{\varepsilon}{3}\right)^M \right\} \in F(I_3)$$

Then $A \cap B \in F(I_3).$ Let $(n,l,k) \in A \cap B.$

Then $\bar{d}(Y_i, Y_j) \leq \bar{d}(Y_i, \Delta_m X_{nlk}{}^{(i)}) + \bar{d}(\Delta_m X_{nlk}{}^{(i)}, \Delta_m X_{nlk}{}^{(j)}) + \bar{d}(\Delta_m X_{nlk}{}^{(j)}, Y_j)$

$$< \varepsilon, \text{ for all } \; i, j \geq n_0.$$

Hence $\langle Y_I \rangle$ is a Cauchy sequence fuzzy real numbers. So there exists a fuzzy real number Y such that $\lim_{i \to \infty} Y_i = Y.$

Let $\eta > 0$ be given. Since $X^{(i)} \to X,$ so there exists $t \in N$ such that

$$\rho\left(X^{(t)}, X\right) < \left(\frac{\eta}{3}\right)^{\frac{1}{\mathrm{M}}} \tag{1}$$

The number t can be chosen in such a way that together with (1) we get

$$\left[\bar{d}(Y_t, Y)\right]^{p_{nLk}} < \frac{\eta}{3}.$$

Since $\left\langle X_{nlk}{}^{(t)} \right\rangle$ is *I*-convergent to Y_t so we have

$$C = \left\{(n, l, k) \in N \times N \times N : \left[\bar{d}(\Delta_m X_{nlk}{}^{(t)}, Y_i)\right]^{p_{nlk}} < \frac{\eta}{3}\right\} \in F(I_3).$$

So for each $(n, l, k) \in C,$ we have

$$\left[\bar{d}(\Delta_m X_{nlk}, Y)\right]^{p_{nlk}} \leq D^2\left[\bar{d}(\Delta_m X_{nlk}, \Delta_{\mathrm{m}} X_{nlk}{}^{(t)})\right]^{p_{nlk}} + D^2\left[\bar{d}(\Delta_m X_{nlk}{}^{(t)}, Y_t)\right]^{p_{nlk}} + D\left[\bar{d}(Y_t, Y)\right]^{p_{nlk}}$$

$$\leq D^2\left(\frac{\eta}{3}\right) + D^2\left(\frac{\eta}{3}\right) + D\left(\frac{\eta}{3}\right) = \eta' \text{ (say),}$$

where $D = \max(1, 2^{H-1}),\quad H = \sup_{n,k} p_{nk} < \infty.$

Hence $\langle \Delta_m X_{nlk} \rangle$ is *I*-convergent to *Y*.

This implies that ${}_3(c^{I(F)})^{BP}(\Delta_m, p)$ is complete.

Using similar technique the other cases can be established.

Theorem 3. *The classes of sequences* ${}_3(c^{I(F)})(\Delta_m, p), {}_3(c_0^{I(F)})(\Delta_m, p), {}_3(c^{I(F)})^R(\Delta_m, p),$ ${}_3(c^{I(F)})^{BP}(\Delta_m, p), {}_3(c_0{}^{I(F)})^{BP}(\Delta_m, p), {}_3(c^{I(F)})^{BR}(\Delta_m, p)$ *are closed under addition and scalar multiplication*

Proof. – We consider the space ${}_3(c_0^{I(F)})(\Delta_m, p)$. Let $\langle X_{nlk} \rangle, \langle Y_{nlk} \rangle \in {}_3(c_0^{I(F)})(\Delta_m, p)$ and $0 < \varepsilon < 1.$ Let α, β be two scalars. Then there exists X_0 and $Y_0 \in R(L)$ be such that

$I_3 - \lim\left[\bar{d}\left(\Delta_m X_{nlk}, X_0\right)\right]^{p_{nlk}} = 0$ and $I_3 - \lim\left[\bar{d}\left(\Delta_m Y_{nlk}, Y_0\right)\right]^{p_{nlk}} = 0$. Then

$$\left[\overline{d}\left(\Delta_m X_{nlk} + \Delta_m Y_{nlk}\ , X_0 + Y_0\right)\right]^{p_{nlk}} \leq D\left[\overline{d}\left(\Delta_m X_{nlk}\ , X_0\right)\right]^{p_{nlk}} + D\left[\overline{d}\left(\Delta_m Y_{nlk}, Y_0\right)\right]^{p_{nlk}}$$

Where $D = \max\left(1, 2^{H-1}\right)$ and H = $\sup\ p_{nlk} < \infty.$

Taking *I*- limits on both sides of the above inequality, we have

$$I_3 - \lim\left[\overline{d}\left(\Delta_m X_{nlk} + \Delta_m Y_{nlk}\ , X_0 + Y_0\right)\right]^{p_{nlk}}$$
$$\leq I_3\ - \lim D\left[\overline{d}\left(\Delta_m X_{nlk}\ , X_0\right)\right]^{p_{nlk}} + I_3\ - \lim D\left[\overline{d}\left(\Delta_m Y_{nlk}, Y_0\right)\right]^{p_{nlk}}$$
$$I_3 - \lim\left[\overline{d}\left(\Delta_m X_{nlk} + \Delta_m Y_{nlk}\ , X_0 + Y_0\right)\right]^{p_{nlk}} = 0.$$

$$\therefore \Delta_m X_{nlk} + \Delta_m Y_{nlk} \in {}_3(c_0^{I(F)})(\Delta_m, p).$$

$\therefore$ the space ${}_3(c_0^{I(F)})(\Delta_m, p)$ is closed under addition. Similarly we can show that ${}_3(c_0^{I(F)})(\Delta_m, p)$ is closed under scalar multiplication.

Theorem 4. *The classes of sequences* ${}_3(c_0^{I(F)})(\Delta_m\ , p),\ {}_3(c_0^{I(F)})^R(\Delta_m, p),\quad {}_3(c_0^{I(F)})^{BP}(\Delta_m, p)$ *and* ${}_3(c_0^{I(F)})^{BR}(\Delta_m\ , p)$ *are solid as well as monotone.*

Proof. We consider the space ${}_3(c_0^{I(F)})(\Delta_m\ , p).$

Let $\langle X_{nlk}\rangle \in {}_3(c_0^{I(F)})(\Delta_m\ , p)$ and $\langle Y_{nlk}\rangle$ be such that $\overline{d}(\Delta_m Y_{nlk}\ , \overline{0}) \leq \overline{d}(\Delta_m X_{nlk}\ , \overline{0}),$ for all $n\, , l, k \in N.$

Let $\varepsilon > 0$ be given. Then the solidness of ${}_3(c_0^{I(F)})(\Delta_m\ , p).$ follows from the following inclusion relation:

$$\{(n\, , l, k) \in N \times N \times N : \left[\overline{d}(\Delta_m X_{nlk}\ , \overline{0})\right]^{p_{nlk}} \geq \varepsilon\} \supseteq \{(n\, , l, k) \in N \times N \times N : \left[\overline{d}(\Delta_m Y_{nlk}\ , \overline{0})\right]^{p_{nlk}} \geq \varepsilon\}.$$

Also by **Lemma 1.** it follows that the classes of sequences under consideration are monotone Using similar technique the other cases can be established.. ■

Theorem 5. *The classes of sequences* ${}_3(c^{I(F)})(\Delta_m\ , p),{}_3\,(c^{I(F)})^R(\Delta_m\ , p),\ {}_3(c^{I(F)})^{BP}(\Delta_m\ , p)$ *and* ${}_3(c^{I(F)})^{BR}(\Delta_m\ , p)$ *are neither solid nor monotone in general*

Proof. The result follows from the following example.

Example 1. Let $A \in I_3$, $p_{nlk} = 1,$ for all $n\, , l, k \in N.$

We consider the sequence $\langle X_{nlk}\rangle$ defined by:

For all $(n, l, k) \notin A,$

$$X_{nlk}(t) = \begin{cases} 1+3nlkt, & \text{for } -\frac{1}{3nlk} \le t \le 0 \\ 1-3nlkt, & \text{for } 0 < t \le \frac{1}{3nlk} \\ 0, & \text{otherwise} \end{cases}$$

otherwise $X_{nlk} = \bar{0}$.

Let m = 3, then we have

$$\Delta_3 X_{nlk}(t) = \begin{cases} 1+\frac{3nlk(n+3)(l+3)(k+3)}{3(nl+lk+nk)+9(n+l+k)+27}t, & \text{for } -\frac{1}{3nlk}+\frac{1}{3(n+3)(l+3)(k+3)} \le t \le 0 \\ 1-\frac{3nlk(n+3)(l+3)(k+3)}{3(nl+lk+nk)+9(n+l+k)+27}t, & \text{for } 0 < t \le \frac{1}{3nlk}-\frac{1}{3(n+3)(l+3)(k+3)} \\ 0, & \text{otherwise} \end{cases}$$

Then $\langle X_{nlk}\rangle \in Z,$ for

$Z = {}_3(c^{I(F)})(\Delta_3 , p),\ {}_3(c^{I(F)})^R(\Delta_3 , p),\ {}_3(c^{I(F)})^{BP}(\Delta_3 , p)$ and ${}_3(c^{I(F)})^{BR}(\Delta_3 , p)$.

Let $K = \{(i, j, k): i + j + k = 3q : q \in N\}$.

We consider the sequence $\langle Y_{nlk}\rangle$ defined by:

$$Y_{nlk} = \begin{cases} X_{nlk}, & \text{if } (n,l,k) \in K, \\ \bar{0}, & \text{otherwise} \end{cases}$$

Then $\langle Y_{nlk}\rangle$ belongs to the canonical pre-image of *K* step space of *Z*,

for $Z = {}_3(c^{I(F)})(\Delta_3 , p),\ {}_3(c^{I(F)})^R(\Delta_3, p),\ {}_3(c^{I(F)})^{BP}(\Delta_3, p)$ and ${}_3(c^{I(F)})^{BR}(\Delta_3, p)$.

But $\langle Y_{nlk}\rangle \notin Z,$ for

$Z = {}_3(c^{I(F)})(\Delta_3, p),\ {}_3(c^{I(F)})^R(\Delta_3, p),\ {}_3(c^{I(F)})^{BP}(\Delta_3, p)$ and ${}_3(c^{I(F)})^{BR}(\Delta_3, p)$.

Hence the class of sequences under consideration are not monotone.

Therefore by ***Lemma 1.*** the class of sequences are not solid. ■

Theorem 6. *The classes of sequences* ${}_3(c_0^{I(F)})^R(\Delta_m, p)$ *and* ${}_3(c_0^{I(F)})^{BR}(\Delta_m, p)$ *are symmetric for* $I_3 = I_3(P)$ *and* $I = I(f),$ *the class of finite subsets of N, otherwise they are not symmetric.*

Proof. Let $I_3 = I_3(P)$ and $I = I(f).$ Then

${}_3(c_0^{I(F)})^R(\Delta_m, p) = {}_3(c_0^F)^R(\Delta_m, p)$ and ${}_3(c_0^{I(F)})^{BR}(\Delta_m, p) = {}_3(c_0^F)^{BR}(\Delta_m, p).$

Now for $\langle X_{nlk}\rangle \in {}_3(c_0^{I(F)})^{BR}(\Delta_m, p)\ (\subset {}_3(c_0^F)^R(\Delta_m, p))$ and given $\varepsilon > 0,$ we have $\{(n,l,k)\in N\times N\times N : \left[\bar{d}(\Delta_m X_{nlk}, X_0)\right]^{p_{nlk}} \geq \varepsilon\}$ is a finite subset of $N\times N\times N.$

Hence the classes of sequences ${}_3(c_0^{I(F)})^R(\Delta_m, p)$ *and* ${}_3(c_0^{I(F)})^{BR}(\Delta_m, p)$ are symmetric.

Theorem 7. *The classes of sequences* ${}_3(c^{I(F)})(\Delta_m, p), {}_3(c_0^{I(F)})(\Delta_m, p), {}_3(c^{I(F)})^R(\Delta_m, p),$ ${}_3(c^{I(F)})^{BP}(\Delta_m, p),$ ${}_3(c_0^{I(F)})^{BP}(\Delta_m, p)$ *and* ${}_3(c^{I(F)})^{BR}(\Delta_m, p)$ *are not symmetric in general.*

Proof. The proof follows from the following example.

Example 2. Let $I_3 = I_3(\rho)$ and $p_{nlk} = \begin{cases} 2, & \text{for } n \text{ even and all } l,k\in N \\ 3, & \text{otherwise} \end{cases}$

We consider the sequence $\langle X_{nlk}\rangle$ defined by:

For $n = i^3,\ i \in N$ and for all $l,k \in N.$

$$X_{nlk}(t) = \begin{cases} 1+\dfrac{t}{3\sqrt[3]{n}}, & \text{for } -3\sqrt[3]{n} \leq t \leq 0 \\ 1-\dfrac{t}{3\sqrt[3]{n}}, & \text{for } 0 < t \leq 3\sqrt[3]{n} \\ 0, & \text{otherwise} \end{cases}$$

otherwise $X_{nlk} = \bar{0}.$

Let m = 3, then we have

$$\Delta_3 X_{nlk}(t) = \begin{cases} 1+\dfrac{t}{3\sqrt[3]{n}-3\sqrt[3]{n+3}}, & \text{for } -3\sqrt[3]{n}+3\sqrt[3]{n+3} \leq t \leq 0 \\ 1-\dfrac{t}{3\sqrt[3]{n}-3\sqrt[3]{n+3}}, & \text{for } 0 < t \leq 3\sqrt[3]{n}-3\sqrt[3]{n+3} \\ 0, & \text{otherwise} \end{cases}$$

Then $\langle X_{nlk}\rangle \in Z,$ for

$$Z = {}_3(c^{I(F)})(\Delta_3, p), {}_3(c_0^{I(F)})(\Delta_3, p), {}_3(c^{I(F)})^R(\Delta_3, p), {}_3(c^{I(F)})^{BP}(\Delta_3, p),$$

$$, {}_3(c_0^{I(F)})^{BP}(\Delta_3, p), {}_3(c^{I(F)})^{BR}(\Delta_3, p).$$

We consider the rearrangement $\langle Y_{nlk}\rangle$ of $\langle X_{nlk}\rangle$ defined by:

For *l*, *k* odd and for all $n \in N,$

$$Y_{nlk}(t) = \begin{cases} 1+\dfrac{t}{3n}, & \text{for } -3n \le t \le 0 \\ 1-\dfrac{t}{3n}, & \text{for } 0 < t \le 3n \\ 0, & \text{otherwise} \end{cases}$$

otherwise $Y_{nlk} = \bar{0}.$

$$\Delta_3 Y_{nlk}(t) = \begin{cases} 1+\dfrac{t}{9}, & \text{for } -9 \le t \le 0 \\ 1-\dfrac{t}{9}, & \text{for } 0 < t \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Then $\langle Y_{nlk} \rangle \notin Z,$ for

$$Z = {}_3(c^{I(F)})(\Delta_3, p), {}_3(c_0^{I(F)})(\Delta_3, p), {}_3(c^{I(F)})^R(\Delta_3, p), {}_3(c^{I(F)})^{BP}(\Delta_3, p),$$

$$, {}_3(c_0^{I(F)})^{BP}(\Delta_3, p), {}_3(c^{I(F)})^{BR}(\Delta_3, p).$$

Hence the classes of sequences under consideration are not symmetric.

Theorem 8. *The classes of sequence* ${}_3(c^{I(F)})(\Delta_m, p), {}_3(c_0^{I(F)})(\Delta_m, p), {}_3(c^{I(F)})^R(\Delta_m, p),$ ${}_3(c_0^{I(F)})^R(\Delta_m, p)$, ${}_3(c^{I(F)})^{BP}(\Delta_m, p), {}_3(c_0{}^{I(F)})^{BP}(\Delta_m, p), {}_3(c^{I(F)})^{BR}(\Delta_m, p)$ $and\, {}_3(c_0^{I(F)})^{BR}(\Delta_m, p)$ *are not convergence free*.

Proof . The result follows from the following example.

Example 3. Let $A \in I_3$, $p_{nlk} = \dfrac{1}{3}$ for all $n, l, k \in N.$

Consider the sequence (X_{nk}) defined as follows:

For all $(n,l,k) \notin A,$

$$X_{nlk}(t) = \begin{cases} 1+3(n+l+k)\,t, & \text{for } \dfrac{1}{3(n+l+k)} \le t \le 0 \\ 1-3(n+l+k)\,t, & \text{for } 0 < t \le \dfrac{1}{3(n+l+k)} \\ 0, & \text{otherwise} \end{cases}$$

Otherwise $X_{nlk} = \bar{0}.$

Let m = 3, then we have,

$$\Delta_3 X_{nlk}(t)=\begin{cases} 1+\frac{3(n+l+k)(n+l+k+9)}{27}t, & \text{for } -\frac{1}{3(n+l+k)}+\frac{1}{3(n+l+k+9)}\le t\le 0 \\ 1-\frac{3(n+l+k)(n+l+k+9)}{27}t, & \text{for } 0<t\le\frac{1}{3(n+l+k)}-\frac{1}{3(n+l+k+9)} \\ 0, & \text{otherwise} \end{cases}$$

Then $\langle X_{nlk}\rangle \in Z,$ for $Z={}_3(c^{I(F)})(\Delta_3,p),{}_3(c_0^{I(F)})(\Delta_3,p),{}_3(c^{I(F)})^R(\Delta_3,p),{}_3(c_0^{I(F)})^R(p),$

${}_3(c^{I(F)})^{BP}(\Delta_3,p),{}_3({c_0}^{I(F)})^{BP}(\Delta_3,p),{}_3(c^{I(F)})^{BR}(\Delta_3,p)\ and {}_3(c_0^{I(F)})^{BR}(\Delta_3,p).$

Consider the sequence $\langle Y_{nlk}\rangle$ defined by:

For $(n,l,k)\notin A,$

$$Y_{nlk}(t)=\begin{cases} 1+\frac{3t}{n+l+k}, & \text{for } -\frac{n+l+k}{3}\le t\le 0 \\ 1-\frac{3t}{n+l+k}, & \text{for } 0<t\le\frac{n+l+k}{3} \\ 0, & \text{otherwise} \end{cases}$$

otherwise $Y_{nlk}=\bar{0}$.

$$\Delta_3 Y_{nlk}(t)=\begin{cases} 1+\frac{3t}{9}, & \text{for } -\frac{n+l+k}{3}-\frac{n+l+k+9}{3}\le t\le 0 \\ 1-\frac{3t}{9}, & \text{for } 0<t\le\frac{n+l+k}{3}+\frac{n+l+k+9}{3} \\ 0, & \text{otherwise} \end{cases}$$

Then $\langle Y_{nlk}\rangle \notin Z,$ for

$Z={}_3(c^{I(F)})(\Delta_3,p),{}_3(c_0^{I(F)})(\Delta_3,p),{}_3(c^{I(F)})^R(\Delta_3,p),{}_3(c_0^{I(F)})^R(\Delta_3,p),$

${}_3(c^{I(F)})^{BP}(\Delta_3,p),{}_3({c_0}^{I(F)})^{BP}(\Delta_3,p),{}_3(c^{I(F)})^{BR}(\Delta_3,p)\ and {}_3(c_0^{I(F)})^{BR}(\Delta_3,p).$

Hence the classes of sequences are not convergence free. ■

Theorem 9. *The classes of sequence* ${}_3(c^{I(F)})(\Delta_m,p),{}_3(c_0^{I(F)})(\Delta_m,p),{}_3(c^{I(F)})^R(\Delta_m,p),$

${}_3(c_0^{I(F)})^R(\Delta_m,p)$, ${}_3(c^{I(F)})^{BP}(\Delta_m,p),{}_3({c_0}^{I(F)})^{BP}(\Delta_m,p),{}_3(c^{I(F)})^{BR}(\Delta_m,p)$
$and {}_3(c_0^{I(F)})^{BR}(\Delta_m,p)$ *are sequence algebras.*

Proof. We consider the space ${}_3(c_0^{I(F)})(\Delta_m,p).$

Let $\langle X_{nlk}\rangle,\langle Y_{nlk}\rangle\in {}_3(c_0^{I(F)})(\Delta_m,p)$ and $0<\varepsilon<1.$

Then the result follows from the following inclusion relation:

$$\left\{(n,k)\in N\times N:\left[\bar{d}(\Delta_m X_{nk}\otimes\Delta_m Y_{nk},\ \bar{0})\right]^{p_{nk}}<\varepsilon\right\}$$

$$\supset\left\{(n,k)\in N\times N:\left[\bar{d}(\Delta_m X_{nk},\bar{0})\right]^{p_{nk}}<\varepsilon\right\}\cap\left\{(n,k)\in N\times N:\left[\bar{d}(\Delta_m Y_{nk},\bar{0})\right]^{p_{nk}}<\varepsilon\right\}.$$

Using similar technique the other cases can be established. ∎

Theorem 10. *The classes of sequence* ${}_3(c^{I(F)})^{BP}(\Delta_m,p), {}_3(c_0{}^{I(F)})^{BP}(\Delta_m,p), {}_3(c^{I(F)})^{BP}(\Delta_m,p)$ *and* $(c_0^{I(F)})^{BR}(\Delta_m,p)$ *are nowhere dense subsets of* ${}_3(\ell_\infty^F)(\Delta_m,p).$

Proof. Since the classes of sequence ${}_3(c^{I(F)})^{BP}(\Delta_m,p), {}_3(c_0{}^{I(F)})^{BP}(\Delta_m,p), {}_3(c^{I(F)})^{BP}(\Delta_m,p)$ *and* $(c_0^{I(F)})^{BR}(\Delta_m,p)$ are proper subsets of the space ${}_3(\ell_\infty^F)(\Delta_m,p).$ we get the result from

Theorem 2.

Theorem 11. *For two sequences* $p=\langle p_{nkl}\rangle$ and $q=\langle q_{nkl}\rangle$ we have

${}_3(c_0{}^{I(F)})^{BP}(\Delta_m,p)\supseteq{}_3(c_0{}^{I(F)})^{BP}(\Delta_m,q)$ *if and only if* $\liminf\limits_{(n,l,k)\in K}\left(\dfrac{p_{nlk}}{q_{nlk}}\right)>0,$ *where* $K\in F(I_3).$

Proof. Let us suppose $\liminf\limits_{(n,l,k)\in K}\left(\dfrac{p_{nlk}}{q_{nlk}}\right)>0$ (1)

Then there exists $\alpha>0$ such that $p_{nlk}>\alpha q_{nlk}$ for sufficiently large pair $(n,\ l,k)\in K.$

Let $\langle X_{nlk}\rangle\in{}_3(c_0^{I(F)})^{BP}(\Delta_m,p).$ Then for $0<\varepsilon<1,$ we have

$A=\{(n,l,k)\in N\times N\times N:\left[\bar{d}(\Delta_m X_{nlk},\bar{0})\right]^{q_{nlk}}<\varepsilon\}\in F(I_3).$

Let $B=K\cap A.$ Then $B\in F(I_3).$

Now for each $(n,l,k)\in B,$ we have $\left[\bar{d}(\Delta_m X_{nlk},\bar{0})\right]^{p_{nlk}}\le\left\{\left[\bar{d}(\Delta_m X_{nlk},\bar{0})\right]^{q_{nlk}}\right\}^{\alpha}.$

Hence $\langle X_{nlk}\rangle\in{}_3(c_0^{I(F)})^{BP}(\Delta_m,q).$

So ${}_3(c_0{}^{I(F)})^{BP}(\Delta_m,p)\supseteq{}_3(c_0{}^{I(F)})^{BP}(\Delta_m,q).$

Next let ${}_3(c_0{}^{I(F)})^{BP}(\Delta_m,p)\supseteq{}_3(c_0{}^{I(F)})^{BP}(\Delta_m,q).$ But there exists no $K\in F(I_3)$ such that (1) holds.

Then there exists a set $C=\{(n_i,l_j,k_m):\ i,j,m\in N\}\subset N\times N\times N$ with $C\notin I_3$ such that

$ip_{n_il_jk_m}<q_{n_il_jk_m}.$

Let the sequence $\langle X_{nlk}\rangle$ be defined by:

$$X_{nlk} = \begin{cases} \left(\bar{\frac{1}{i}}\right)^{\frac{1}{q_{n_i k_j}}}, \text{ if } & n = n_i, l = l_j, k = k_m \\ \bar{0}, & \text{otherwise} \end{cases}$$

Then $\langle X_{nlk} \rangle \in {}_3(c_0^{I(F)})^{BP}(\Delta_m, p).$

But $\left[\bar{d}(\Delta_m X_{n_i l_j k_m}, \bar{0})\right]^{p_{n_i l_j k_m}} > \exp\left(\frac{-\log i}{i}\right),$ which is a contradiction.

Corollary. *For two sequences* $p = \langle p_{nkl} \rangle$ and $q = \langle q_{nkl} \rangle$, *we have*

$${}_3(c_0^{I(F)})^{BP}(\Delta_m, p) = {}_3(c_0^{I(F)})^{BP}(\Delta_m, q) \quad \textit{if} \quad \liminf_{(n,l,k)\in K}\left(\frac{p_{nlk}}{q_{nlk}}\right) > 0 \quad \textit{and} \quad \liminf_{(n,l,k)\in K}\left(\frac{q_{nlk}}{p_{nlk}}\right) > 0$$

where $K \in F(I_2)$.

Proof. The result easily follows from the **Theorem 10.**

IV.CONCLUSION

In this paper we have introduced and studied the notion of Ideal convergent difference multiple sequences spaces of fuzzy real numbers. We have established the completeness property of the introduced class of sequences. We have verified some algebraic and topological properties. The introduced notion can be applied for further investigations from different aspects.

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