Again, about a Leontovich's equation

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Abstract - A new version of a solution of the Leontovich's integral equation for a current in a linear radiator, accomplished in a form of a thin-wall straight metal cylinder with circular cross section, is considered. The solution is based on using the method of variation of constants and is reduced to the sequential calculating the terms of the series for the current, located in powers of the small parameter of the thin antennas' theory.
The terms of the series are obtained in the form of an integral expressions that facilitates their comparison with other known solutions. The results of calculating the input impedances of several antennas variants are given.

***Keywords* – Antenna theory, Integral equation, Thin linear radiator.**

I. INTRODUCTION

As is known, a thin straight metal rod is one of the main types of radiators. It is widely used in practice as an independent antenna and as an element of a complex antenna. The calculation method of a complex antenna can be based on its division into more simple elements. Therefore, the theory of a direct metal radiator occupies an important place in antennas theory. The rigorous method of calculating such a radiator depends on solving the integral equation for the current. Knowing the current distribution $J(z)$ along the radiator, one can compute its electromagnetic field $E_z(J)$ and all electrical characteristics of the antenna.

Consider the symmetric radiator with an arm length *L*, located along an axis of a cylindrical coordinate system. The radiator has the simplest form. Its cross section has the shape of a circle with a radius *a*. In the radiator center, which coincides with the origin of the coordinate system, an extraneous (external) field $K(z)$ is included (Fig.1).

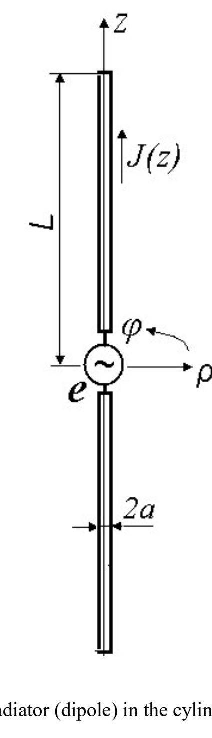


Figure 1. Symmetric radiator (dipole) in the cylindrical coordinate system

A field created by the radiator must satisfy a boundary condition on the its metal surface with the length $2L$ and the radius *a*:

$$E_z(J)\Big|_{\rho=a,-L\le z\le L} + K(z) = 0 \; . \quad (1)$$

A current distribution must satisfy a condition of the current absence at the radiator ends:

$$J(\pm L) = 0. \quad (2)$$

An expression (1) is a mathematical representation of a known fact that the resulting field on the ideally conducting surface of the radiator, which is a sum of the extraneous field and the field created by the radiator current, must be zero. The extraneous field is usually specified as δ -function:

$$K(z) = e\delta(z-h). \quad (3)$$

Here *e* is the potentials difference between a gap edges in the radiator center, *h* is the gap coordinate. Solving equation (1), one can find the generator current, and hence an input impedance of the antenna.

Expression (1) contains as an embryo all the integral equations of the theory of a direct metal radiator. An external appearance of the equations is determined mainly by the choice of function $E_z(J)$. In this case, electric currents parallel to the *z* axis create electromagnetic fields, i.e. the corresponding component of the field strength is

$$E_z(J) = -\frac{\omega}{k^2}\left(k^2 A_z + \frac{\partial^2 A_z}{\partial z^2}\right) , (4)$$

where $A_z(\rho,z) = \dfrac{\mu}{4\pi}\int\limits_{-L}^{L} J(\varsigma)\int\limits_{0}^{2\pi} G d\varphi d\varsigma$ is the vector potential, ω is the circular frequency, *k* is a propagation constant in the free space, μ is the magnetic permeability, G is the Green function. Using these expressions, one can obtain the Hallen's integral equation for the current along a straightthin-wall metal cylinder (an equation with an exact kernel). Using equality $A_z(\rho,z) = \dfrac{\mu}{4\pi}\int\limits_{-L}^{L} J(\varsigma) G d\varsigma$ for the vector potential, we obtain the Hallen's equation for the current along the filament [1].

The equation for the current along a filament of a finite radius (an equation with an approximate kernel) has also become widespread. The first solution of this equation was given by Hallen himself and described in detail in [2]. In this solution, the method of successive approximations (iterative procedure) is applied. As a parameter, in the inverse powers of which the function $J(z)$ is expanded into a series, the value $\Omega = 2\ln(2L/a)$ is used. More accurate results is given by the iterative process proposed by R. King and D. Middleton [3]. In it, the expansion parameter Ω was replaced by Ψ .

The Leontovich's equation [4] played a large and, unfortunately, underestimated role in the development of the thin antennas' theory. This article presents new results obtained by means of solving this equation.

II.DERIVATION OF THE LEONTOVICH'SEQUATION AND THE METHOD OF ITS SOLUTION

In [4], the radiator model in the form of a thin-wall straight metal cylinder with circular cross-section was considered. The vector potential ***A***of this model field has only a component A_z, which in accordance with the above expression is equal to

$$A_z(\rho,z) = \frac{\mu}{8\pi^2}\int\limits_{0}^{2\pi} T(z,\varphi)d\varphi \; , (5)$$

where $T(z,\varphi) = \int\limits_{-L}^{L} J(\varsigma)\dfrac{\exp(-jkR)}{R}d\varsigma$, $R = \sqrt{(z-\varsigma)^2+\rho_1^2}$, $\rho_1 = \sqrt{\rho^2 + a^2 - 2a\rho\cos\varphi}$, $k = \omega\sqrt{\mu\varepsilon}$.

Integrating $T(z,\varphi)$ by parts and consistently using a circumstance that the radiator radius a is small compared to its length and wavelength, i.e. neglecting the terms of the order a/L and ka andkeeping the terms proportional to the logarithm of these quantities, one can find

$$T(z,\varphi) = -2J(z)\ln p\rho_1 - \int\limits_{-L}^{L}\exp(-jk|\varsigma - z|)sign(\varsigma - z)\ln 2p|\varsigma - z|\left[\frac{dJ(\varsigma)}{d\varsigma} - jkJ(\varsigma)sign(\varsigma - z)\right]d\varsigma \cdot$$

Here *p* is some constant, having the dimensions of inverse length, and $\rho_1 > \rho > a$, i.e.,

$$\int_0^{2\pi} \ln p\rho_1 d\varphi = \int_0^{2\pi} \ln p\sqrt{\rho^2 + a^2 - 2a\rho\cos\varphi}\, d\varphi = 2\pi \ln p\rho \cdot$$

It means then near the antenna

$$A_z(\rho, z) = \frac{\mu}{4\pi}\left[-2J(z)\ln p\rho + V(J,z)\right], \qquad (6)$$

where $V(J,z) = \int_{-L}^{L} \exp(-jk|z-\varsigma|)\ln 2p|z-\varsigma|\left[jkJ(\varsigma) + sign(z-\varsigma)\frac{dJ(\varsigma)}{d\varsigma}\right]d\varsigma \cdot$

Substituting (6) into (4) and assuming that $\rho = a$, we'll find the tangent component of the antenna's electric field:

$$E_z(a,z) = \frac{1}{4\pi j\omega\varepsilon}\left[\chi^{-1}\left(\frac{d^2J}{dz^2} + k^2J\right) + \frac{d^2V}{dz^2} + k^2V\right]. \qquad (7)$$

The magnitude $\chi = -1/(2\ln pa)$ is named a small parameter of the thin antenna theory. As is shown in [5], in the capacity of magnitude $1/p$ one should choose the distance to the nearest inhomogeneity, i.e. the smallest of three values : wavelength λ, antenna length $2L$, and a radius R_c of its curvature. For a straight radiator, the length of which does not exceed the wavelength, it is expedient to choose the magnitude

$$\chi = 1/\left[2\ln(2L/a)\right] = 1/\Omega \; . (8)$$

Let introduce the notation $4\pi j\omega\varepsilon W(J,z) = \frac{d^2V}{dz^2} + k^2V$. Using it, we obtain from (1) and (7) the desired equation

$$\frac{d^2J}{dz^2} + k^2J = -4\pi j\omega\varepsilon\chi\left[K(z) + W(J,z)\right] \cdot (9)$$

In square brackets of this equation, along with the exciting extraneous emf, an integral term depending on the current distribution in the conductor is located. That is an additional emf included in the antenna wire, caused by radiation and distributed along the antenna wire. This emf allows to provide a sinusoidal current along the antenna, which, as is shown in [6], otherwise, using a single external emf, one cannot to create.

The goal of the transformations, which were performed in the derivation of equation (9), is separating the logarithmic singularity. In contrast to the original integral (5), the functions A_z and $W(J,z)$ involved in expressions (6) and (9) are continuous functions. This circumstance is an importantadvantage of the equation (9). The second virtue of this equation is the absence of an argument φ in it. It shows that the integration with respect to φ was performed. However, this equation is valid for a tubular antenna, i.e. it is equivalent to the Hallen's equation not with an approximate, but with an exact kernel. Therefore, when using this equation in the course of calculations, one can assume that the currents are concentrated on the radiator's axis. That greatly simplifies the calculations. At the same time, the accuracy of the order adopted in the derivation of the equation is preserved.

To solve equation (9) in [4], the perturbation method was used, i.e. the solution is sought as a series in powers of a small parameter χ :

$$J(z) = J_0(z) + \chi J_1(z) + \chi^2 J_2(z) + \ldots \; .(10)$$

Substituting the series (10) into equation (9) and equating the coefficients with equal powers of χ, in the case of an untuned radiator, when $J_0(z) = 0$, we arrive to the system of equations:

$$\frac{d^2J_1}{dz^2} + k^2J_1 = -4\pi j\omega\varepsilon K(z),\; J_1(\pm L) = 0,$$

$$\frac{d^2J_n}{dz^2} + k^2J_n = -4\pi j\omega\varepsilon W(J_{n-1}),\; J_n(\pm L) = 0,\; n > 1. \qquad (11)$$

In [4] the system of equations (11) was solved in the second approximation, and the expression for the current in the antenna is givenat the point $z = 0$. If by means of this method to calculate the current at an arbitrary point of the antenna, then for $\chi = 1/[2\ln(2L/a)]$ (the authors use a small parameter equal to $\chi_1 = 1/[2\ln(1/ka)]$), we get [7]:

$$J(z) = j\chi J_1(z) + j\chi^2 J_2(z), \quad (12)$$

where $J_1(z) = \frac{e}{60\cos\alpha}\sin(\alpha - |t|)$, $J_2(z) = \frac{e\Theta(t,\alpha)}{120\sin 2\alpha\cos\alpha}$, $\Theta(t,\alpha) =$

$$\sin(\alpha+t)\{C + \ln 2(\alpha - t) - Ei[-j2(\alpha-t)] + e^{j2\alpha}\langle Ei(-j4\alpha) - Ei[-j2(\alpha+t)]\rangle + e^{-j2\alpha}\ln[(\alpha+t)/(2\alpha)]\} +$$

$$\sin(\alpha-t)\{C + \ln 2(\alpha + t) - Ei[-j2(\alpha+t)] + e^{j2\alpha}\langle Ei(-j4\alpha) - Ei[-j2(\alpha-t)]\rangle + e^{j2\alpha}\ln[(\alpha-t)/(2\alpha)]\} +$$

$$2\cos\alpha\sin(\alpha+t)\{e^{j\alpha}[Ei(-j2\alpha) - Ei(-j2t)] + e^{-j\alpha}\ln(\alpha/t)\} -$$

$$2\cos\alpha\sin(\alpha-t)\left\{e^{j\alpha}[Ei(-j2\alpha) - Ei(-j2t)] + e^{-j\alpha}\ln\frac{\alpha}{t}\right\}, Ei(jx) = Cix + jSix, t = kz, \alpha = kL.$$

It is easy to verify that in the first approximation the current is distributed in a sinusoidal law. The input impedance of the antenna in the same approximation is purely reactive and equal to

$$Z_{A1} = -j60\chi^{-1}\cot\alpha \quad , (13)$$

that coincides with the input impedance of an equivalent long line, the wave impedance of which is equal to $W = 60/\chi$. The active component appears in the second approximation. As was shown in [4], where the tuned and untuned radiators are first considered separately, the expression for the current in the untuned radiator has the general nature and can be used in both cases.

III. INTEGRAL EXPRESSION FOR THE CURRENT AND INPUT IMPEDANCE

In [4] the calculation resultsfor the current and the input impedance of the direct metal radiatorare given in the form of anaggregate of tabulated functions. This circumstance made difficult to compare the results obtained by means of thesolvingLeontovich's integral equation with the results obtained by means of another known methods, in particular, by the induced emf method.

The book [8] describes another solution of the equations system(11), first published in [9]. When $n > 1$,if to consider that the values $W(J_{n-1})$ are known, one may employ the method of variation of constants. Using the perturbation method, when the solution is sought as a series in powers of a small parameter χ, we put that the functional $W(J,z)$ is linear, i.e. $W(J,z) = \sum_{n=1}^{\infty}\chi^n W(J_n)$ .As a result, we get:

$$\chi^n J_n(z) = j\frac{\chi}{30\sin 2\alpha}\{\sin k(L+z)\int_z^L W(\chi^{n-1}J_{n-1})\sin k(L-\varsigma)d\varsigma + \sin k(L-z)\int_{-L}^z W(\chi^{n-1}J_{n-1})\sin k(L+\varsigma)d\varsigma \ .(14)$$

In the particular case of a symmetric radiator with the extraneous emf in the antenna center

$$j\chi^n J_n(z) = j\frac{\chi}{30\sin 2\alpha}\{\sin k(L-|z|)\int_{-L}^{L} W(j\chi^{n-1}J_{n-1})\sin k(L-|\varsigma|)d\varsigma \ . \quad (15)$$

Expressions (14) and (15) allow us in the general case and in the case of a symmetric radiator with an external emf located in the radiator center to find the n-th term of the series (10) for the current and, accordingly, the n-th approximation, if the (n-1)-th approximation is known. As follows from the above expressions, in order to obtain a current $J(z)$ at an arbitrary point z of the radiator, it is need to multiply the corresponding value of the input current $J(0)$ at its upper and lower arms by $\sin k(L-|z|)/\sin\alpha$.

In accordance with equality (7), we write expressions for the electric fields of individual currents components on the radiator surface, with allowance for the different order of their smallness. When $n > 1$, we get

$$E_z(j\chi^n J_n) = -W(j\chi^{n-1}J_{n-1}) + W(j\chi^n J_n) \ . \quad (16)$$

Here the field $-W\left(j\chi^{n-1}J_{n-1}\right)$ created by the (n-1)-th component of the current is the negative external emf for the n-th component of the current. This n-th component of the current creates an extraneous emf $W\left(j\chi^{n}J_{n}\right)$ for the (n+1)-th component of the current. Therefore, there is no reason to take n-th component into account as part of the n-th current, i.e. $E_z\left(j\chi^{n}J_{n}\right)=-W\left(j\chi^{n-1}J_{n-1}\right)$, and equality (15) should be replaced by

$$j\chi^{n}J_{n}(z)=j\frac{\chi}{30\sin 2\alpha}\sin k\left(L-|z|\right)\int_{-L}^{L}E_{\varsigma}\left(j\chi^{n-1}J_{n-1}\right)\sin k\left(L-|\varsigma|\right)d\varsigma \,. \quad (17)$$

In particular, when $n=2$,

$$j\chi^{2}J_{2}(z)=j\frac{\chi}{30\sin 2\alpha}\sin k\left(L-|z|\right)\int_{-L}^{L}E_{\varsigma}\left(j\chi J_{1}\right)\sin k\left(L-|\varsigma|\right)d\varsigma \,. \quad (18)$$

When n=1,

$$j\chi J_{1}(z)=j\frac{\chi}{30\sin 2\alpha}\sin k\left(L-|z|\right)\int_{-L}^{L}K(\varsigma)\sin k\left(L-|\varsigma|\right)d\varsigma = ej\chi J_{1}(0)\sin k\left(L-|z|\right) . \quad (19)$$

Comparing expressions (18) and (19), it is easy to be convinced that at the point $z=0$ the second term $j\chi^{2}J_{2}(0)=\int_{-L}^{L}E_{\varsigma}\left(j\chi J_{1}\right)\chi J_{1}(\varsigma)d\varsigma$ of the series for the current is proportional to the square of the first term of this series. The value of the integral in this expression with different degrees of accuracy was repeatedly calculated during the determination of the input impedance of a linear radiator by the induced emf method. It is equal to

$$\int_{-L}^{L}E_{\varsigma}\left(j\chi J_{1}\right)\chi J_{1}(\varsigma)d\varsigma=-\chi^{2}J_{1}^{2}(0)Z_{A} \,. \quad (20)$$

From this expression it follows that the first and second terms of the series (10) have opposite signs, and the magnitude of the second term is equal to the magnitude of the first one multiplied by a constant $N=-\chi J_{1}(0)Z_{A}$.

Expression (17) allows us to successively go from the (n-1)-th summand to the n-th. The transition is facilitated by the structure similarity of the integral terms with different n. These terms differ from each other only by constant coefficients of proportionality, which are equal to N. For example, during transition to the third current component the value $E_{\varsigma}\left(j\chi J_{1}\right)$ in the integrand of expression (20) will be replaced by the value $E_{\varsigma}\left(j\chi J_{2}\right)$ multiplied by N.Thisis easy seenfrom (18). Thus, each new term of the series for the currentis equal to the previous one, multiplied by the same value.

As follows from the obtained results, all components of the current are sinusoidal. The value of the n-th component is equal to

$$\chi^{n}J_{n}(z)=\chi^{n-1}N^{n-1}\cdot j\chi J_{1}(z) \,. \quad (21)$$

Each next component is different from the previous by sign. Since the quantity entering into expression (20) is complex, the total current and the input impedance of the antenna, starting from the second term, become complex quantities.

Using (18) and (19) one can come to the expression that follows from (12) in the particular case - for a symmetric radiator with anemf located in the radiator center. If to replace (18) and (19) by expression (14) for $n=2$ corresponding to an asymmetric radiator, it is easy to arrive at (12).

IV. ABOUTTHE METHOD OF INDUCED EMF

The input impedance of the radiator in the second approximation is equal to

$$Z_{A2}=e/J_{A}(0)\,, \quad (22)$$

where $J_{A}(0)=j\chi J_{1}(0)+j\chi^{2}J_{2}(0)$ is the antenna input current. If

$$J_{A}(0)\neq 0\,, \quad (23)$$

then putting that $\chi^2 J_2(0) \ll \chi J_1(0)$, we find

$$Z_{A2} = \frac{e}{j\chi J_1(0)\left\{1 + j\chi^2 J_2(0)/\left[j\chi J_1(0)\right]\right\}} = \frac{e\left[j\chi J_1(0) - j\chi^2 J_2(0)\right]}{\left[j\chi J_1(0)\right]^2} \quad (24)$$

This expression, when substituting integral formulas (18) and (19) into it, completely coincides with the second formulation of the induced emf method. Anidentity of the integral expressions explains the well-known fact of coincidence of the input impedances, if they are expressed through tabulated functions, and also of their numerical values, obtained by means of the Leontovich's equation and by the induced emf method.

In the vicinity of the parallel resonance, where condition (23) is not satisfied, the transition from the expression (23) to the expression (24) is inapplicable. Therefore, the method of induced emf in the vicinity of a parallel resonance gives an incorrect result (the input impedance increases without limit), while the input impedance calculated by formula (22), i.e. in accordance with the solution of the Leontovich's equation, remains by finite magnitude.

The first formulation of the induced emf method is founded on the equality of the reactive power given off by the source of the emf and the reactive power passing through a closed surface surrounding radiator. It is characterized by the presence of an asterisk, symbolizing complex conjugate values. It is known that the reactive power appeared by misunderstanding, its physical meaning was widely denied everywhere, but this did not prevent from itsusing for calculations and conclusions. In the end, the point of view presented in [10] won. It replaced this non-existent quantity by real oscillating power, reflecting the existence and change of instantaneous power. In antennas technique the use of the first formulation led to the fact that all losses resistances (in substances and from the skin effect) turned out to be negative, and the results of calculations of electrical characteristics are unstable.

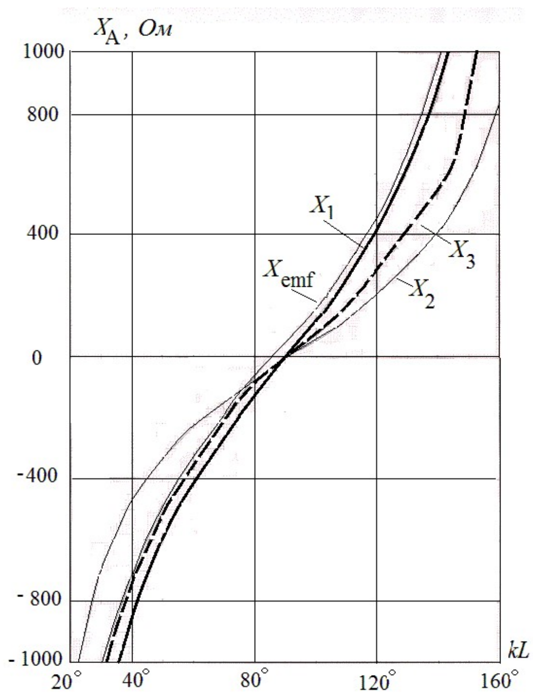
The second formulation of the method of induced emf was proposed by M.I. Kontorovich on the basis of the principle of reciprocity [10]. But it is also easily derived from the energy ratios, if to equate the oscillating power given off by the emf source (generator) and the oscillating power passing through a closed surface surrounding the radiator. The coincidence of the integral obtained by solving the Leontovich's equation with an integral based on the induced emf method showed that the coincidence of the numerical results of both methods is not accidental.

It should be emphasized that two current components defined by expressions (19) and (18) are calculated as a result of solving the integral equation. Expression (24), obtained by the method of induced emf, is a result of transformations that can hardly be called strict. First, the transition from the factor $1/(1-\Delta)$ to the factor $(1+\Delta)$ was accomplished. In this case, from general considerations it is considered that $\Delta << 1$. But it is difficult to calculatethe magnitude of the error caused by this transition, depending on the parameters of the structure. In the denominator of expression (24) there are factors that are zero at the frequency of a parallel resonance. Therefore, in the field of the parallel resonance the result of using the induced emf method is far from true. In this frequency range the use of expressions (18) and (19) is clearly preferable.

It is interesting to compare several options of the input impedances magnitudes, calculated with varying degrees of accuracy. The first option -the impedance $Z_1 = e/\left[j\chi J_{A1}(0)\right]$ - is based on the first term of the series for the current. The second option is the impedance $Z_2 = e/\left[j\chi J_1(0) + j\chi^2 J_2(0)\right]$, considering the first two terms of the series (10) and calculated in accordance with (22). The third option includes the three terms in accordance with (21). Finally, one more impedance Z_{emf} is given in accordance with the method of induced emf. The results of reactive components calculations (all four variants) are given in Fig. 2 for radiator with a magnitude $L/a = 200$ (χ is equal to 0.0834) and in Fig. 3 for the longer and thinner antenna with a magnitude $L/a = 2000$ (χ =0.603).

The Figures show that the curve X_2 is significantly different from X_1, and the curve X_3 partially returns to the curve X_1. The value X_{emf} is different from all previous ones. The reduction of a small parameter in the area of positive reactances brings the curves closer to each other. In Fig. 4 the curves are given for the resistances: in Fig. 4a - for antenna with $L/a = 200$, in Fig. 4b - for antenna with $L/a = 2000$. It can be seen from the

Figures that accounting for several terms of the series (10) significantly changes the results, and the transition from expressions (18) and (19) to the expression (24), used in the method of induced emf, distorts them.



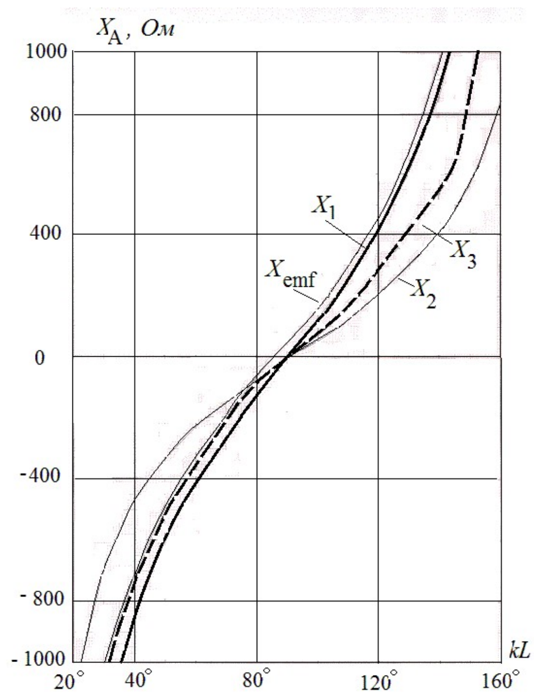


Figure 2. Reactive impedance of antenna with $\frac{L}{a} = 200$ Figure 3. Reactive impedance of antenna with $\frac{L}{a} = 2000$

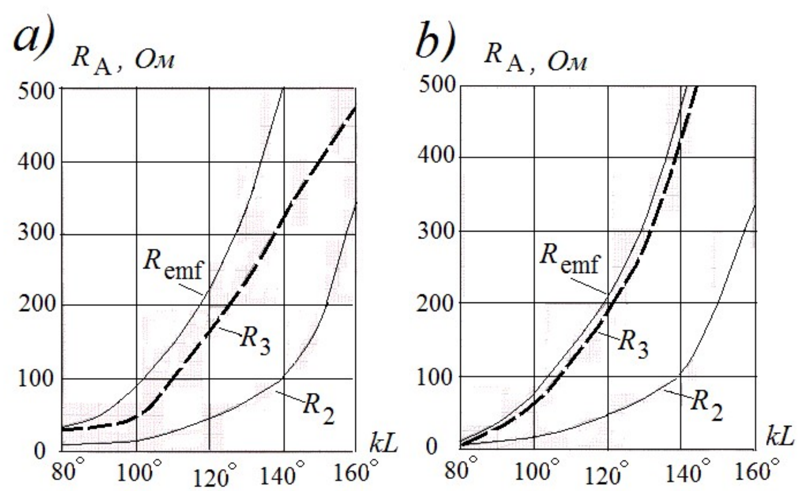


Figure 4. Active impedance of antenna with $\frac{L}{a} = 200$ (a) and $\frac{L}{a} = 2000$ (b)

A new variant of solving the Leontovich's integral equation for the current in a linear radiator allows us to obtain new results and significantly deepen our understanding of the processes in a linear antenna.

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