Plane Partitions and n Color Partitions

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Abstract: In this paper, we give an overview of plane partitions and partitions with n copies of n. We also explain a bijection between these two types of partitions using Agarwal's version of Bender and Knuth bijection (Bender and Knuth, J. Combin. Theory (A), 13, 1972, 40-54). Using this bijection, we show how self-conjugate partitions with n copies of n gives us a new class of plane partitions. We also obtain the corresponding associated lattice paths.

I. Introduction

The theory of partitions has an interesting history. Certain problems in partitions certainly date back to middle ages. The foundation of partitions was laid by Leonard Euler in 18th century. Many other renowned mathematicians Cayley, Gauss, Jacobi, Schur and MacMahon later worked in this area. Plane partitions are generalization of ordinary partitions of integers introduced by P.A. MacMahon [9]. On the other hand partitions with n copies of n were defined by Agarwal and Andrews. Since both plane partitions and partitions with n copies of n have same generating function, there is scope of direct bijection between these two types of partitions.

Various tools have been used from time to time to study the plane partitions. One of these is Schensted correspondence between matrices and plane partitions which was extended by Knuth and later used by Bender and Knuth [2]. They used this correspondence to prove some new results. Much later, Agarwal established a correspondence between Bender Knuth matrices and partitions with n copies of n. This lead to a double bijection between plane partitions and partitions with n copies of n.

Agarwal Bender-Knuth bijection is a great tool to study plane partitions with the help of advancements in the study of partitions with n copies of n. Many known results for partitions with n copies of n have been interpreted for plane partitions using this bijection.

II. Basic Definitions

The word partition has numerous meanings in Mathematics. Anytime a division of some object into sub objects is undertaken, the word partition is likely to pop up. Formally, we define a partition of a positive integer as follows:

Definition 2.1 Partition: (Euler) A partition λ of a positive integer ν is defined as a finite sequence of positive integers $\lambda_1, \lambda_2, ... \lambda_r$ arranged in non-increasing order $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_r$ such that $\nu = \lambda_1 + \lambda_2 + ... + \lambda_r$ where $\lambda_1, \lambda_2, ..., \lambda_r$ are called parts of the partition λ of ν.
For example, 3+2+1 is a partition of 6.

Definition 2.2 Plane Partition: Plane Partition (Macmahon, [9]). A plane partition of a positive integer ν is an array of non-negative integers.

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \ldots & \ldots & \ldots & \end{matrix}$$

$$\text{for which } \sum_{i,j} n_{ij} = \nu$$

and rows and columns are in non-increasing order.

Definition 2.2 Shape of a plane partition [11]: If in a plane partition π of a positive integer ν, there are λ_i parts in the i^{th} row of π so that, for some r, $\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_r > \lambda_{r+1} = 0$, then we call the partition $\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_r$ of the integer $p = \lambda_1 + \lambda_2 + \lambda_3 ... + \lambda_r$, the shape of π.

If the non-zero entries of a plane partition π are strictly decreasing in each column (row), we say that π is a column (row) strict plane partition.

If in a plane partition π, $(i,j)^{th}$ entry is same as the $(j,i)^{th}$ $\forall$ i, j, it is called a symmetric plane partition.

Definition2.3 Partitions with n copies of n [5]. Partitions with n copies of n (also called *n*-colour partitions) introduced by Agarwal and Andrews in 1987 are defined as follows:

A partition with n copies of n (also called n-colour partition) is a partition in which a part of size n, $n \geq 0$ can come in n different colors denoted by subscripts: $n_1, n_2, ... n_n$.

For example, Partitions of 3 with n copies of n are: 3_1, 3_2, 3_3, $2_2 + 1_1$, $2_1 + 1_1$, $1_1 + 1_1 + 1_1$

Definition2.4 Conjugate of an *n*- colour partition [7]. Let $\pi = (a_1)_{b1} + (a_2)_{b2} + \cdots + (a_r)_{br}$ be an n-colour partition of ν. We call $(a_i)_{ai-bi+1}$ the conjugate of $(a_i)_{bi}$.

An n-colour partition of ν obtained from π by replacing each of its parts by its conjugate will be called the conjugate of π and is denoted by π^c.

For example, $3_3 1_1$ is conjugate of the partition $3_1 1_1$ of the integer 4.

Definition2.5 Self Conjugate Partition. An n-colour partition π is said to be self-conjugate if it is identical with its conjugate π^c.

In [5], it was shown that if $P(\nu)$ denotes the number of n-colour partitions of ν, then

$$1 + \sum_{\nu=1}^{\infty} P(\nu) q^{\nu} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}$$

Since the right-hand side of above equation is the generating function for the plane partitions also, this implies that the number of n-colour partitions of ν equals the number of plane partitions of ν.

III. BIJECTION BETWEEN PLANE PARTITIONS AND N-COLOUR PARTITIONS

3.1 Bender Knuth Bijection

In the 1960s, a correspondence between certain matrices and plane partitions was developed by Schensted and later extended by Knuth. This combinatorial mapping allowed a great simplification in the deduction of many known restricted plane partition generating functions. Furthermore, many new generating functions can be treated using this process.

Agarwal [3] established a bijection $\psi.\ \varphi$ between n-color partitions and plane partitions. For the clarity of our presentation, we shall first reproduce the bijection $\psi.\varphi$ here. In $\psi.\varphi$, φ is due to Bender and Knuth [8] and is the 1-1 correspondence of the following theorem:

Theorem3.1.1 There is one to one correspondence between

1. the set of k x k matrices with non-negative integer entries
2. the set of all lexicographically ordered sequences of ordered pairs of integers, each $\leq k$
3. the set of ordered pairs (π_1, π_2) of column strict plane partitions of same shape in which each entry doesn't exceed k.

Note that the correspondence described in Theorem 3.1 produces a one-to-one correspondence between symmetric matrices A of non-negative integers and plane partitions π with strict decrease along columns.

A different version of this theorem is also found in literature (cf. Stanley [12, 7.20]).

Theorem (Bender and Knuth). There is one to one correspondence between plane partitions of ν, on the one hand, and infinite matrices $a_{i,j}$ ($i,j \geq 1$) of non-negative integer entries which satisfy

$$\sum_{r \geq 1} \left\{ \sum_{i+j=r+1} a_{ij} \right\} = \nu$$

on the other.

In the sequel, we shall call images $\varphi(\pi)$ BK_{ν}- matrices (B for Bender and K for Knuth).

Although, these matrices are infinite matrices, but we represent them by largest possible square matrices containing at least one non-zero entry in the last row (or in the last column).

Thus, for example, we'll represent six BK_3- matrices by

$$3, \quad \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}, \quad \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}, \quad \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}, \quad \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}, \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$$

3.2 Agarwal's Double Bijection

Definition3.2.1 (Agarwal) Define a matrix $E_{i,j}$ as an infinite matrix whose $(i, j)^{th}$ entry is 1 and the other entries are all zeros. We call $E_{i,j}$ distinct parts of a BK_v- matrix.

Now we define the mapping ψ as follows:

Let $\Delta = a_{11}E_{1,1} + a_{12}E_{1,2} + \ldots + a_{21}E_{2,1} + a_{22}E_{2,2} + \ldots$

be a BK_v- matrix where a_{ij} are non-negative integers which denote the multiplicities of $E_{i,j}$.

We map each part $E_{p,q}$ of Δ to a single part m_i of an n-colour partition of v. The map denoted by ψ is defined as

$$\psi: E_{p,q} \to (p + q - 1)_p, \qquad (3.2.1)$$

and the inverse mapping ψ^{-1} is easily seen to be

$$\psi^{-1}: m_i \to E_{i,\, m-i+1}. \qquad (3.2.2)$$

Under this mapping, we see that each BK_v matrix uniquely corresponds to an n-colour partition of v and vice versa.

The composite of two mappings φ and ψ denoted by $\psi.\varphi$ is clearly a bijection between plane partitions of v on one hand, and n-colour partitions of v, on the other.

For example,

n-colour partitions of 4	BK4 Matrices	Pair of Plane Partitions	Corres. Plane partition of 4
41	E1,4	4, 1	4
42	E2,3	3, 2	3 1
43	E3,2	2, 3	2 1 1
44	E4,1	1, 4	1 1 1 1
3111	E1,3 + E1,1	3 1, 1 1	3 1
3211	E2,2 + E1,1	2 1, 2 1	2 1 1
3311	E3,1 + E1,1	1 1, 3 1	1 1 1 1
2121	E1,2 + E1,2	2 2, 1 1	2 2
2221	E2,1 + E1,2	2 , 2 1 1	2 2
2222	E2,1 + E2,1	1 1, 2 2	1 1 1 1
211111	E1,2 + E1,1 + E1,1	2 1 1, 1 1 1	2 1 1
221111	E2,1 + E1,1 + E1,1	1 1 1, 2 1 1	1 1 1 1
11111111	E1,1 + E1,1 + E1,1 + E1,1	1 1 1 1, 1 1 1 1	1 1 1 1

We shall use this double bijection to between plane partitions and partitions with n copies of n to prove the results in the next section.

Next, we recall the following description of associated lattice paths defined by Anand and Agarwal [4].

All paths are of finite length lying in the first quadrant. They will begin on the *y*- axis and terminate on the *x*- axis. Only three moves are allowed at each step.

northeast: from (i, j) to $(i + 1, j + 1)$,

southeast: from (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$,

horizontal: from (i, j) to $(i + 1, j)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.

The following terminology will be used in describing associated lattice paths:

Truncated Isosceles Trapezoidal Section (TITS): A section of path which starts on *x*-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on *x*-axis forms what we call a Truncated Isosceles Trapezoidal Section.

Since the lower base lies on *x*-axis and is not a part of the path, hence the term truncated.

Slant Section (SS): A section of path consisting of only southeast steps which starts on the *y*-axis (origin not included) and ends on the *x*-axis.

Height of a slant section is '*t*' if it starts from $(0,t)$. Clearly, a path can have an SS only in the beginning.

A lattice path can have at most one SS.

Weight of a TITS: To define this, we shall represent every TITS by an ordered pair $\{a,b\}$ where a denotes its altitude and b the length of the upper base.

Weight of the TITS with ordered pair $\{a,b\}$ is a units.

Weight of a lattice path is the sum of weights of its TITSs.

Slant Section is assigned weight zero.

For example, in Figure 1, the associated lattice path has one SS of height 1 and one TITS with ordered pair {2, 3} and its weight is 2 units.

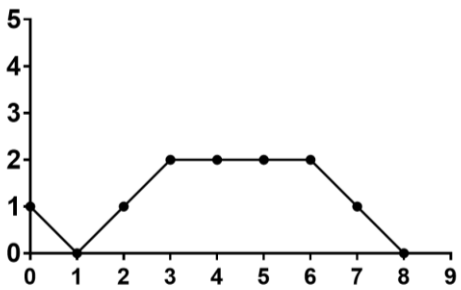

Fig: 1

IV. A NEW CLASS OF SELF-CONJUGATE N-COLOUR PARTITIONS

We shall prove the following result:

Theorem 4.1.1 Let $P(v)$ denote the number of self-conjugate partitions of v with n copies of n of the form

$$\sum_i (a_i)_{b_i}$$

such that $a_1 > a_2 > a_3 > \ldots\ldots\ldots > a_r$.

Let Let $Q(v)$ denote the number of plane partitions of v of the form

$$\pi = \begin{matrix} l_{11} & l_{12} & \cdots \\ l_{21} & l_{22} & \cdots \\ \vdots & \vdots & \end{matrix}$$

such that

a) $l_{1j} = b_j \ \forall\ 1 \leq j \leq r$ and
b) $l_{ij} = 1 \ \forall\ 2 \leq i \leq b_j$ and $\forall\ 1 \leq j \leq r$

Then, $P(v) = Q(v)\ \forall\ v$.

For example, for $v = 6$, $P(6) = 1$, as the only self-conjugate n-colour partition enumerated by $P(6)$ is $5_3 1_1$

Also, $Q(6) = 1$, as the only plane partition enumerated by $Q(6)$ is $\begin{matrix} 3 & 1 \\ 1 & \\ 1 & \end{matrix}$

Hence, $P(6) = Q(6) = 1$

Proof: Let $w = (a_1)_{b1} + (a_2)_{b2} + \cdots + (a_r)_{br}$ be an n-colour partition enumerated by $P(v)$, that is,

$a_1 > a_2 > a_3 > \ldots\ldots\ldots > a_r$.

Now as w is a self-conjugate partition with all parts distinct. Thus, each part $(a_i)_{bi} = (a_i)_{ai - bi + 1}$. Therefore, corresponding BK_v- matrix is

$$\Delta = E_{b_1,a_1-b_1+1} + E_{b_2,a_2-b_2+1} + \cdots + E_{b_r,a_r-b_r+1}$$

i.e.

$$\Delta = E_{b_1,b_1} + E_{b_2,b_2} + \cdots + E_{b_r,b_r},$$

Also, $b_i = a_i - b_i + 1$

i.e. $2b_i = a_i + 1$ which implies $b_i = (a_i + 1)/2$.

Now as $a_1 > a_2 > a_3 > \ldots\ldots\ldots > a_r$

Hence, $(a_1 + 1)/2 > (a_2 + 1)/2 > (a_3 + 1)/2 > \ldots\ldots\ldots > (a_r + 1)/2$

i.e. $b_1 > b_2 > b_3 > \ldots\ldots\ldots > b_r$.

This BK_v- matrix corresponds to pair of sequences

$$\begin{matrix} b_1 & b_2 & \ldots & b_r \\ b_1 & b_2 & \ldots & b_r \end{matrix}$$

This pair of sequences corresponds to pair of column strict plane partitions (π_1, π_2) of same shape. The lower sequence corresponds to π_1 and upper sequence corresponds to π_2. Since the lower sequence is non-increasing,

$\pi_1 = b_1\ b_2\ \ldots\ b_r$

and π_2 is of same shape. So $\pi_2 = b_1\ b_2 \ldots b_r$.

This pair (π_1, π_2) corresponds to plane partition

$$\pi = \begin{matrix} l_{11} & l_{12} & \cdots \\ l_{21} & l_{22} & \cdots \\ \vdots & \vdots & \end{matrix}$$

where $l_{1j} = b_j \ \forall\ 1 \leq j \leq r$ and $l_{ij} = 1 \ \forall\ 2 \leq i \leq b_j$ and $\forall\ 1 \leq j \leq r$.

Hence, proved.

Corollary4.1: The plane partitions enumerated by $Q(\nu)$ above are such that each column of this partition is a self-conjugate partition considered as ordinary partition.

For example, the partition enumerated by $Q(6)$ is

$$\begin{array}{ll} 3 & 1 \\ 1 & \\ 1 & \end{array}$$

The first column corresponds to ordinary partition 3+1+1 which is a self-conjugate partition as can be seen using Ferrers graph

Also, second column corresponds to ordinary partition 1 which is also a self-conjugate partition.

For the next result, we define the following:

Definition 2.1 The mass of any TITS with ordered pair $\{a, b\}$ is defined to be $a - b$.

Definition 2.1 The base of any TITS with ordered pair $\{a, b\}$ is defined to be b.

The next result gives us three-way correspondence.

Theorem 4.1.2 Let $R(\nu)$ denote the number of associated lattice paths of weight ν such that

(a.) For any TITS with ordered pair $\{a,b\}$, b doesn't exceed a.

(b.) TITS are arranged in order of non-decreasing weights.

(c.) there is no SS

(d.) if weight of a TITS is greater than the other TITS, then its base is greater than the base of the other.

(e.) Base of each TITS is one greater than its mass.

Then $P(\nu)= Q(\nu) = R(\nu)\ \forall\ \nu$.

Proof of the Theorem: Consider an n-colour partition enumerated by $P(\nu)$. Each part a_b of n-colour partition corresponds to a TITS with ordered pair {a, b}. As in n color partition b can't exceed a hence, for any TITS with ordered pair $\{a, b\}$, b doesn't exceed a.

Since n-colour partition is self-conjugate, therefore, $b_i = a_i - b_i + 1$for each part $(a_i)_{b_i}$

So, in the corresponding TITS, we get that the base is one greater than its mass.

This gives a one-to-one correspondence between partitions enumerated by $P(\nu)$ and associated lattice paths enumerated by $R(\nu)$.

Combining with previous theorem we establish a three-way identity $P(\nu)= Q(\nu) = R(\nu)\ \forall\ \nu$.

Hence, proved.

V. CONCLUSION

Theorem 4.1.1 gives us a direct correspondence between a class of column wise, self-conjugate plane partitions of an integer ν and a class of self-conjugate partitions with n copies of n of the integer ν. Also, since partitions with n copies of n have a lattice path representation. So, we obtain a lattice path representation for a class of restricted plane partitions. We hope to interpret other definitions of conjugacy of plane partitions to n- colour partitions.

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