A three-way bijection between class of associated lattice paths, plane partitions and partitions with n copies of n[1]

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Abstract. In this paper, a three-way bijection between Associated Lattice paths (Anand and Agarwal, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No.1, February 2012, 23-39), a class of plane partitions and a class of partitions with n copies of n is established. We shall use the correspondence between associated lattice paths and partitions with n copies of n and Agarwal's version of Bender and Knuth bijection (Bender and Knuth, J. Combin. Theory (A), 13, 1972, 40-54) to establish this bijection. As a result, we also get lattice path representation for class of partitions with n copies of n.

Keywords - Bender and Knuth Bijection, Associated lattice paths, colored partitions, partitions with n copies of n, plane partitions.

I. INTRODUCTION

We first recall the definitions and results we shall use in this paper:

Definition1.1(Agarwal and Andrews [5]). A partition with n copies of n (also called n color partition) is a partition in which a part of size n, $n \geq 0$ can come in n different colors denoted by subscripts: $n_1, n_2, \ldots n_n$.

For example, Partitions of 3 with n copies of n are: $3_1, 3_2, 3_3, 2_2 + 1_1, 2_1 + 1_1, 1_1 + 1_1 + 1_1$,

Definition1.2 Plane Partition (Macmahon, [9]). A plane partition of a positive integer ν is an array of non-negative integers.

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \cdots & \cdots & \cdots & \end{matrix}$$

$$\text{for which } \sum_{i,j} n_{ij} = \nu$$

and rows and columns are in non-increasing order.

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If in a plane partition π of a positive integer v, there are λ_i parts in the i^{th} row of π so that, for some r, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_r > \lambda_{r+1} = 0$, then we call the partition $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_r$ of the integer $p = \lambda_1 + \lambda_2 + \lambda_3 \ldots + \lambda_r$, the shape of π.

Next, we recall the following description of associated lattice paths defined by Anand and Agarwal [4].

All paths are of finite length lying in the first quadrant. They will begin on the y- axis and terminate on the x- axis. Only three moves are allowed at each step.

northeast: from (i, j) to $(i + 1, j + 1)$,

southeast: from (i, j) to $(i + 1, j − 1)$, only allowed if $j > 0$,

horizontal: from (i, j) to $(i + 1, j)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.

The following terminology will be used in describing associated lattice paths:

Truncated Isosceles Trapezoidal Section (TITS): A section of path which starts on x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on x-axis forms what we call a Truncated Isosceles Trapezoidal Section.

Since the lower base lies on x- axis and is not a part of the path, hence the term truncated.

Slant Section (SS): A section of path consisting of only southeast steps which starts on the y- axis (origin not included) and ends on the x- axis.

Height of a slant section is 't' if it starts from $(0, t)$. Clearly, a path can have an SS only in the beginning.

A lattice path can have at most one SS.

Weight of a TITS: To define this, we shall represent every TITS by an ordered pair $\{a, b\}$ where a denotes its altitude and b the length of the upper base.

Weight of the TITS with ordered pair $\{a, b\}$ is a units.

Weight of a lattice path is the sum of weights of its TITSs.

Slant Section is assigned weight zero.

For example, in Figure 1, the associated lattice path has one SS of height 1 and one TITS with ordered pair $\{2, 3\}$ and its weight is 2 units.

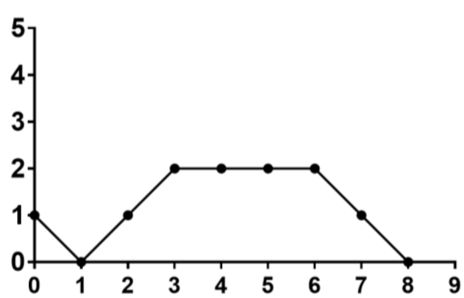


Fig: 1

Agarwal [3] established a bijection $\psi.\ \varphi$ between n- color partitions and plane partitions. For the clarity of our presentation, we shall first reproduce the bijection $\psi.\varphi$ here. In $\psi.\varphi$, φ is due to Bender and Knuth [8] and is the 1- 1 correspondence of the following theorem:

Theorem. There is one to one correspondence between

1. the set of k x k matrices with non-negative integer entries
2. the set of all lexicographically ordered sequences of ordered pairs of integers, each $\leq k$
3. the set of ordered pairs (π_1, π_2) of column strict plane partitions of same shape in which each entry doesn't exceed k.

A different version of this theorem is also found in literature (cf. Stanley [11, 7.20]).

Theorem (Bender and Knuth). There is one to one correspondence between plane partitions of v, on the one hand, and infinite matrices $a_{i,j}$ $(i,j \geq 1)$ of non-negative integer entries which satisfy

$$\sum_{r\geq 1}\left\{\sum_{i+j=r+1} a_{ij}\right\} = v$$

on the other.

In the sequel, we shall call images $\varphi(\pi)$ BK_v- matrices (B for Bender and K for Knuth). Although, these matrices are infinite matrices, but we represent them by largest possible square matrices containing at least one non-zero entry in the last row (or in the last column).

Thus, for example, we'll represent six BK_3- matrices by

$$3\ ,\ \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\ ,\ \begin{matrix}1 & 1\\ 0 & 0\end{matrix}\ ,\ \begin{matrix}0 & 0\\ 0 & 1\end{matrix}\ ,\ \begin{matrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix}\ ,\ \begin{matrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{matrix}$$

Definition (Agarwal). We define a matrix $E_{i,j}$ as an infinite matrix whose $(i, j)^{th}$ entry is 1 and the other entries are all zeros. We call $E_{i,j}$ distinct parts of a BK_v- matrix.

Now we define the mapping ψ as follows:

Let $\Delta = a_{11}E_{1,1} + a_{12}E_{1,2} + \ldots + a_{21}E_{2,1} + a_{22}E_{2,2} + \ldots$

be a BK_v- matrix where a_{ij} are non-negative integers which denote the multiplicities of $E_{i,j}$.

We map each part $E_{p,q}$ of Δ to a single part m_i of an n-color partition of v. The map denoted by ψ is defined as

$$\psi: E_{p,q} \rightarrow (p + q - 1)_p, \qquad (1.2)$$

and the inverse mapping ψ^{-1} is easily seen to be

$$\psi^{-1}: m_i \rightarrow E_{i,\, m-i+1}. \qquad (1.3)$$

Under this mapping, we see that each BK_v matrix uniquely corresponds to an n-color partition of v and vice versa. The composite of two mappings φ and ψ denoted by $\psi.\varphi$ is clearly a bijection between plane partitions of v on one hand, and n-color partitions of v, on the other.

II. ASSOCIATED LATTICE PATHS AND PLANE PARTITIONS

In this section, we establish a bijection between a class of plane partitions and a class of partitions with n copies of n. Also, since partitions with n copies of n are in one to one correspondence with a class of associated lattice paths, we also obtain a lattice path representation for this new class of plane partitions. We shall prove the following result:

Theorem 2.1 Let $P(v)$ denote the number of partitions of v with n copies of n of the form

$$\sum_i (a_i)_{b_i}$$

such that

(a.) $b_1 > b_2 > b_3 > \dots\dots\dots > b_r$

(b.) $a_1 - b_1 < a_2 - b_2 < a_3 - b_3 < \dots\dots\dots < a_r - b_r$

Let $Q(v)$ denote the number of plane partitions of v of the form

$$\pi = \begin{array}{l} l_{11} \\ l_{21} \\ \vdots \end{array}$$

such that $l_{ij}=0\ \forall\ j \geq 2$ and there are exactly b_1 entries in first column of π.

Then, $P(v)= Q(v)\ \forall\ v$.

Example. $P(6) = 11$, since the relevant partitions are

$6_1,\ 6_2,\ 6_3,\ 6_4,\ 6_5,\ 6_6,\ 4_4 + 2_1,\ 2_2 + 4_1,\ 3_2 + 3_1,\ 3_3 + 3_1,\ 3_3 + 3_2$

Also, $Q(6) = 11$, in this case the relevant partitions are

6,	5,	4,	4 ,	3 ,	3 ,	3 ,	2,	2,	2,	1
	1	2	1	3	2	1	2	2	1	1
			1		1	1	2	1	1	1
						1		1	1	1
									1	1
										1

Proof of Theorem 2.1 Let w = $(a_1)_{b1} + (a_2)_{b2} + \cdots + (a_r)_{br}$ be an *n*-color partition enumerated by $P(v)$, that is,

$b_1 > b_2 > b_3 > \ldots\ldots\ldots > b_r$

and

$a_1 - b_1 < a_2 - b_2 < a_3 - b_3 < \ldots\ldots\ldots < a_r - b_r$.

Corresponding BK_v- matrix is

$$\Delta = E_{b_1,a_1-b_1+1} + E_{b_2,a_2-b_2+1} + \cdots + E_{b_r,a_r-b_r+1}$$

where $b_i > b_{i+1}$ and $a_i - b_i + 1 < a_{i+1} - b_{i+1} + 1\ \forall\ 1 \le i \le r-1$.

This corresponds to pair of sequences

$$\begin{matrix} b_1 & b_2 & . & . & . & b_r \\ a_1 - b_1 + 1 & a_2 - b_2 + 1 & . & . & . & a_r - b_r + 1 \end{matrix}$$

This pair of sequences corresponds to pair of column strict plane partitions (π_1, π_2) of same shape. The lower sequence corresponds to π_1 and upper sequence corresponds to π_2. Since the lower sequence is strictly increasing, therefore:

$$\pi_1 = \begin{matrix} a_r - b_r + 1 \\ a_{r-1} - b_{r-1} + 1 \\ . \\ . \\ a_1 - b_1 + 1 \end{matrix} \quad \textit{and} \quad \pi_2 = \begin{matrix} b_1 \\ b_2 \\ . \\ . \\ b_r \end{matrix}$$

Note that π_1 and π_2 are of same shape.

This pair (π_1, π_2) corresponds to plane partition

$$\pi = \begin{matrix} l_{11} \\ l_{21} \\ \vdots \end{matrix}$$

where $l_{11} = a_r - b_r + 1$, $l_{12} = a_{r-1} - b_{r-1} + 2$ and so on

and there are exactly b_1 entries in the column. This plane partition of v is enumerated by $Q(v)$. Hence, the result.

Note. Let w be a partition enumerated by $P(v)$ and let its corresponding plane partition be π. Then

- the number of rows of π *equals* b_1.

- All parts of partition enumerated by $P(v)$ are distinct. If parts are equal, subscripts are distinct.

For the next result, we define the following:

Definition 2.1 The mass of any TITS with ordered pair $\{a, b\}$ is defined to be $a - b$.

Definition 2.1 The base of any TITS with ordered pair $\{a, b\}$ is defined to be b.

In [4], it was proved that partitions of v with n copies of n are in one to one correspondence with associated lattice paths of weight v such that for any TITS with ordered pair $\{a, b\}$, b doesn't exceed a, TITS arranged in order of non-decreasing weights and not having any SS. In the following result, we replace the condition of TITS arranged in order of non-decreasing weights by the condition that TITS are arranged in order of strictly decreasing base and increasing mass. This won't affect the already established bijection as the parts of n color partition are distinct or if parts are equal the have different subscripts.

In view of this result and the result proved above, following result can be proved easily.

Theorem 2.2 Let $C(\nu)$ denote the number of associated lattice paths of weight v such that:

(a.) For any TITS with ordered pair $\{a, b\}$, b doesn't exceed a

(b.) TITS are arranged in order of strictly decreasing base and increasing mass.

(c.) there is no SS

(d.) if base of a TITS is greater than the other TITS, then its mass is smaller than the mass of the other.

(e.) TITS with equal weights cannot have equal mass.

Then $P(\nu)= Q(\nu) = C(\nu)\ \forall\ \nu$.

(a.) Proof: Consider an n color partition enumerated by $P(\nu)$. Each part a_b of n color partition corresponds to a TITS with ordered pair {a, b}. As in n color partition b can't exceed a hence, for any TITS with ordered pair $\{a, b\}$, b doesn't exceed a

. Also, we arrange the parts of n color partition in order of decreasing subscripts. Subscript of each part of n color partition corresponds to the base of corresponding TITS. Therefore, the following condition

$a_1 - b_1 < a_2 - b_2 < a_3 - b_3 < \ldots\ldots\ldots < a_r - b_r$

leads to the condition that if base of a TITS is greater than the other TITS, then its mass is smaller than the mass of the other. Also, two equal parts with same subscript are not allowed, hence the condition that TITS with equal weights cannot have equal mass. This establishes a one to one correspondence between n color partitions enumerated by $P(\nu)$ and Associated lattice paths enumerated by $C(\nu)$.

Also, as $P(\nu)= Q(\nu)\ \forall\ \nu$ by Theorem 2.1

Hence, we establish three-way bijection:

$P(\nu)= Q(\nu) = C(\nu)\ \forall\ \nu$.

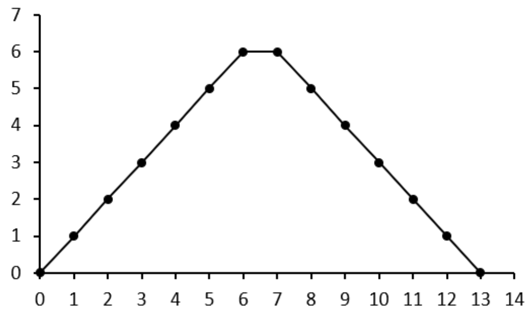
Note: We explain this beautiful three-way bijection with the help of following example.

Example: $P(6) = Q(6) = C(6) = 11$

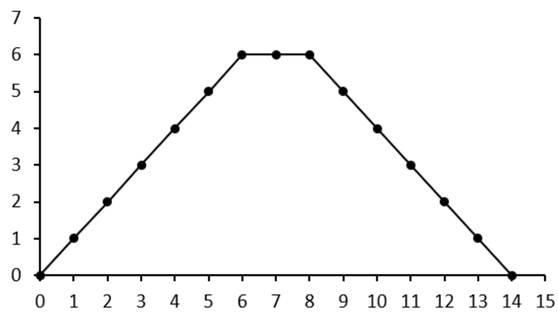
Following series of figures show one to one correspondence between plane partitions enumerated by $Q(6)$, partitions with n copies of n enumerated by $P(6)$ and lattice paths enumerated by $C(6)$

PLANE PART. EN. BY Q (6) ⟷ N COLOR PART. EN. BY P (6) ⟷ ASSOCIATED LATTICE PATHS EN. BY C (6)

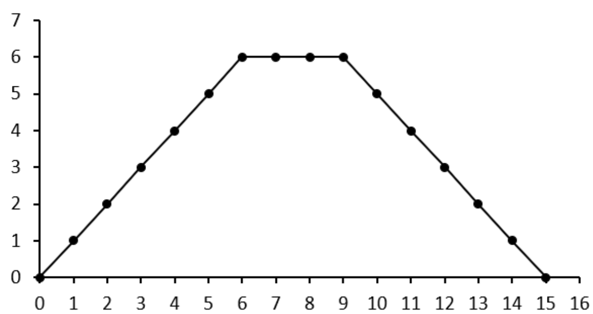
6 ⟷ 6_1 ⟷



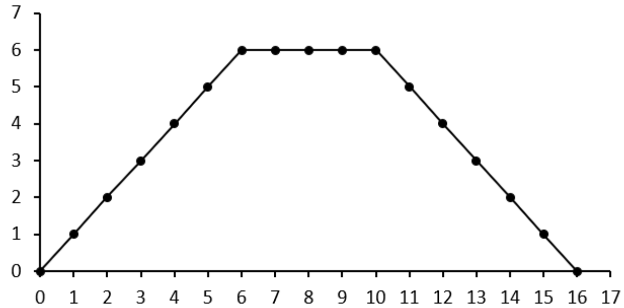
5
1 ⟷ 6_2 ⟷



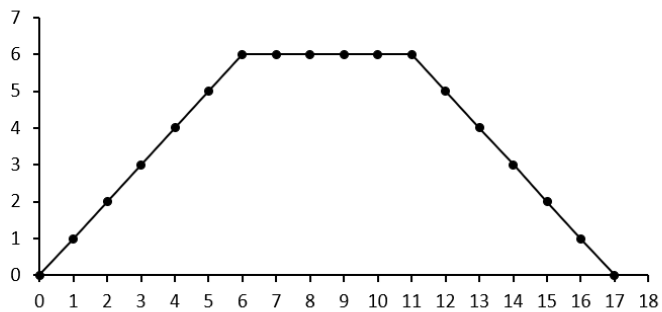
4
1
1 ⟷ 6_3 ⟷



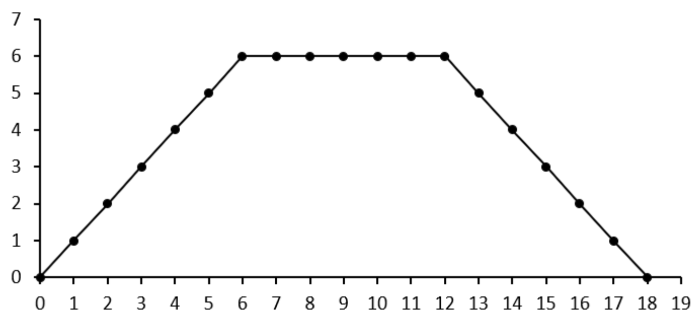
3 ⟷ 6_4 ⟷
1
1
1



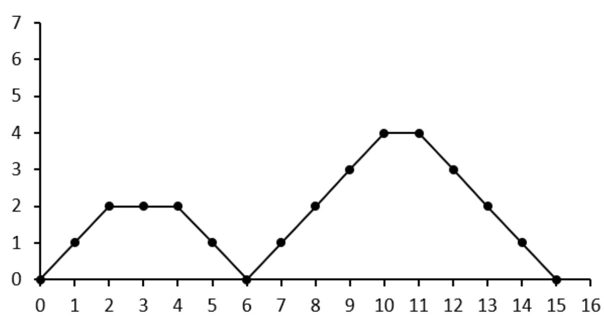
2 ⟷ 6_5 ⟷
1
1
1
1



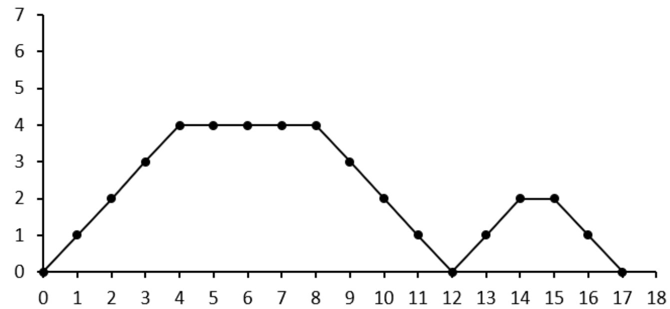
1 ⟷ 6_6 ⟷
1
1
1
1
1



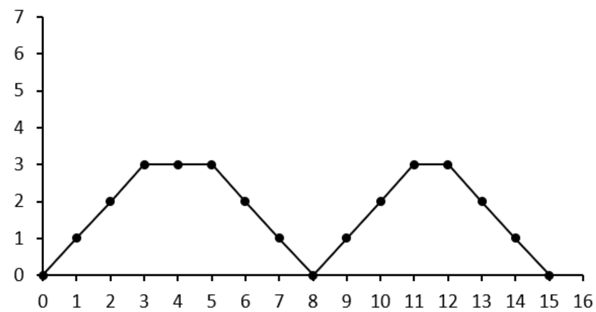
4 ⟷ $2_2 4_1$ ⟷
2



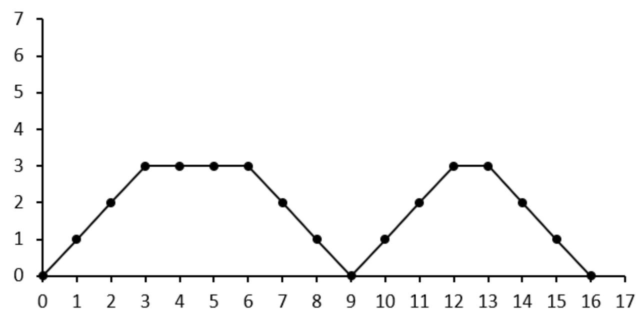
2 ↔ $4_4 2_1$ ↔
2
1
1



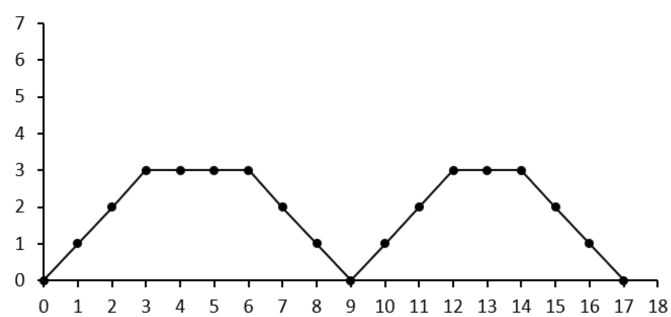
3 ↔ $3_2 3_1$ ↔
3



3 ↔ $3_3 3_1$ ↔
2
1



2 ↔ $3_3 3_2$ ↔
2
2



III. CONCLUSION.

Theorem 2.1 gives us a direct correspondence between a class of plane partitions of an integer *v* and a class of partitions with *n* copies of *n* of the integer *v*. Also, since partitions with *n* copies of *n* have a lattice path representation. So, Theorem 2.2 leads to a three way bijection This three-way bijection gives rise to six identities. Aso, we obtain a lattice path representation for this new class of restricted plane partitions. But there is scope of finding bijection between rest of the plane partitions and n color partitions.

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